Qualitative Studies
ISSN: 1308 724X (NWSAQS)
ID: 2018.13.2.E0040

Status : Original Study
Received: January 2018
Accepted: April 2018

Yunus Bulut
İnönü University, ybulut79@gmail.com, Malatya-Turkey
Mehmet Güngör
İnönü University, mgungor44@gmail.com, Malatya-Turkey

DOI	http://dx.doi.org/10.12739/NWSA.2018.13.2.E0040	
ORCID ID	0000-0002-9108-4937	0000-0001-6869-4043
CORRESPONDING AUTHOR	Yunus Bulut	

ON JOINT DISTRIBUTIONS OF ORDER STATISTICS FROM NONIDENTICALLY DISTRIBUTED DISCRETE VARIABLES

ABSTRACT

In this study, the joint distributions of order statistics arising from *innid* discrete random variables are expressed in the form of an integral by using permanent. Then, the results related to *pf* and *df* distributions are given.

Keywords: Order Statistics, Discrete Random Variable, Probability Function, Distribution Function, Permanent

1. INTRODUCTION

Several identities and recurrence relations for probability density function (*pdf*) and distribution function (*df*) of order statistics of independent and identically distributed (*iid*) random variables were established by numerous authors including Arnold et al. [1], Balasubramanian and Beg [4], David [14], and Reiss [21]. Furthermore, Arnold et al. [1], David [14], Gan and Bain [15], and Khatri [18] obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Balakrishnan [2] showed that several relations and identities that have been derived for order statistics from continuous distributions also hold for the discrete case. Nagaraja [19] explored the behavior of higher order conditional probabilities of order statistics in a attempt to understand the structure of discrete order statistics. Nagaraja [20] considered some results on order statistics of a random sample taken from a discrete population. Corley [12] defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller [16]. Guilbaud [17] expressed the probability of the functions of independent but not necessarily identically distributed (*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West [10]. In addition, Vaughan and Venables [22] derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. Balakrishnan [3], and Bapat and Beg [8] obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents. Using multinomial arguments, the *pdf* of $X_{r:n+1}$ ($1 \le r \le n+1$) was obtained by Childs and Balakrishnan [11] by

How to Cite:
Bulut, Y. and Güngör, M., (2018). On Joint Distributions of Order Statistics from Nonidentically Distributed Discrete Variables, **Qualitative Studies (NWSAQS)**, 13(2):41-48, DOI: 10.12739/NWSA.2018.13.2.E0040.

adding another independent random variable to the original n variables $X_1, X_2, ..., X_n$. Also, Balasubramanian, et al., [7] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. In a paper published in 1991, Beg [9] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer et al. [13] derived the expressions for the distribution and density functions by Ryser's method and the distribution of maxima and minima based on permanents. In the first of two papers, Balasubramanian, et al., [5] obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, ..., X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian et al. [6] generalized their previous results [5] to the case of the joint distribution function of several order statistics.

2. RESEARCH SIGNIFICANCE

In general, the distribution theory for order statistics is complex when the parent distribution is discrete. In this study, the joint distributions of p order statistics of *innid* discrete random variables are expressed in form of an integral. As far as we know, these approaches have not been considered in the framework of order statistics from *innid* discrete random variables.

From now on, the subscripts and superscripts are defined in the first place in which they are used and these definitions will be valid unless they are redefined.

If $\mathrm{a}_1, \mathrm{a}_2, ...$ are defined as column vectors, then the matrix obtained by taking m_1 copies of a_1, m_2 copies of a_2,… can be denoted as $[\underset{m_1}{\mathrm{a}_1} \;\; \underset{m_2}{\mathrm{a}_2} \;\; ...]$ and *per*A denotes the permanent of a square matrix A, which is defined as similar to determinants except that all terms in the expansion have a positive sign.

Let $X_1, X_2, ..., X_n$ be *innid* discrete random variables and $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be the order statistics obtained by arranging the n X_i's in increasing order of magnitude. Let F_i and f_i be *df* and *pf* of X_i (i =1, 2,…, n), respectively.

The *df* and *pf* of $X_{r_1:n}, X_{r_2:n}, ..., X_{r_p:n}$, $0 = r_0 < r_1 < r_2 < ... < r_p < r_{p+1} = n+1$ (p =1, 2,…, n), will be given. For notational convenience we write $\sum\limits_{m_p, k_p ..., m_1, k_1}$, $\sum\limits_{z_p, ..., z_2, z_1}$, $\int$ and $\int\limits_{\mathrm{V}}$ instead of $\sum\limits_{m_p=0}^{n-r_p} \sum\limits_{k_p=0}^{r_p - r_{p-1} - 1} ... \sum\limits_{m_2=0}^{r_3 - r_2 - 1} \sum\limits_{k_2=0}^{r_2 - r_1 - 1} \sum\limits_{m_1=0}^{r_2 - r_1 - 1} \sum\limits_{k_1=0}^{r_1 - 1}$, $\sum\limits_{z_1=0}^{x_1} \sum\limits_{z_2=z_1}^{x_2} \sum\limits_{z_3=z_2}^{x_3} ... \sum\limits_{z_p=z_{p-1}}^{x_p}$, $\int\limits_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)} \int\limits_{F_{\varsigma_4^{(1)}}(x_2-)}^{F_{\varsigma_4^{(1)}}(x_2)} ... \int\limits_{F_{\varsigma_{2p}^{(1)}}(x_p-)}^{F_{\varsigma_{2p}^{(1)}}(x_p)}$ and $\int\limits_{0}^{F_{\varsigma_2^{(1)}}(x_1)} \int\limits_{v_{\varsigma_2^{(1)}}^{(1)}}^{F_{\varsigma_4^{(1)}}(x_2)} ... \int\limits_{v_{\varsigma_{2(p-1)}^{(1)}}^{(p-1)}}^{F_{\varsigma_{2p}^{(1)}}(x_p)}$ in the expressions below, respectively $(x_i = 0,1,2,...)(z_0 = 0)$.

3. THEOREMS FOR DISTRIBUTION AND PROBABILITY FUNCTIONS

In this section, the theorems related to *pf* and *df* of $X_{r_1:n}, X_{r_2:n}, ..., X_{r_p:n}$ will be given. We will now express the following theorem for the joint *pf* of *p* order statistics of *innid* discrete random variables.

Theorem 3.1.

$$f_{r_1,r_2,\dots,r_p:n}(x_1,x_2,\dots,x_p)=\mathrm{D}\sum_{n_{\varsigma_1},n_{\varsigma_2},\dots,n_{\varsigma_{2p}}}\int\left(\prod_{w=1}^{p+1}per[\underset{r_w-r_{w-1}-1}{\mathrm{v}^{(w)}-\mathrm{v}^{(w-1)}}][\varsigma_{2w-1}/.)\right)\prod_{w=1}^{p}per[\underset{1}{\mathrm{dv}^{(w)}}][\varsigma_{2w}/.)\,, \tag{1}$$

where $x_1<x_2<\dots<x_p$, $\mathrm{D}=\prod_{w=1}^{p+1}[(r_w-r_{w-1}-1)!]^{-1}$, $\mathrm{v}^{(w)}=(v_1^{(w)},v_2^{(w)},\dots,v_n^{(w)})'$, $\mathrm{dv}^{(w)}=(dv_1^{(w)},dv_2^{(w)},\dots,dv_n^{(w)})'$, $\mathrm{v}^{(0)}=0=(0,0,\dots,0)'$, and $\mathrm{v}^{(p+1)}=1=(1,1,\dots,1)'$ are column vectors, $\sum_{n_{\varsigma_1},n_{\varsigma_2},\dots,n_{\varsigma_{2p}}}$ denotes the sum over $\bigcup_{\ell=1}^{2p}\varsigma_\ell$ for which $\varsigma_\upsilon\cap\varsigma_\vartheta=\phi$ for $\upsilon\neq\vartheta$, $\bigcup_{\ell=1}^{2p+1}\varsigma_\ell=\{1,2,\dots,n\}$, $v^{(t)}_{\varsigma_{2w-1}^{(i_w)}}=\left[v^{(t)}_{\varsigma_{2t}^{(1)}}-F_{\varsigma_{2t}^{(1)}}(x_t-)\right]\dfrac{f_{\varsigma_{2w-1}^{(i_w)}}(x_t)}{f_{\varsigma_{2t}^{(1)}}(x_t)}+F_{\varsigma_{2w-1}^{(i_w)}}(x_t-)$ and

$$\varsigma_\ell=\begin{cases}\{\varsigma_\ell^{(1)}\}, & \text{if } \ell \text{ even}\\ \{\varsigma_\ell^{(1)},\varsigma_\ell^{(2)},\dots,\varsigma_\ell^{(r_{\frac{\ell+1}{2}}-r_{\frac{\ell-1}{2}}-1)}\}, & \text{if } \ell \text{ odd}\end{cases}.$$

Here, n_{ς_ℓ} is the cardinality of ς_ℓ. $\mathrm{A}[\varsigma_\ell/.)$ is the matrix obtained from A by taking rows whose indices are in ς_ℓ.

Proof. It can be written

$$f_{r_1,r_2,\dots,r_p:n}(x_1,x_2,\dots,x_p)=P\{X_{r_1:n}=x_1,X_{r_2:n}=x_2,\dots,X_{r_p:n}=x_p\}\,. \tag{2}$$

(2) can be expressed as

$$f_{r_1,r_2,\dots,r_p:n}(x_1,x_2,\dots,x_p)=\sum_{m_p,k_p\dots,m_1,k_1}\mathrm{C}\;per[\underset{r_1-1-k_1}{\mathrm{F}(x_1-)}\;\underset{k_1+1+m_1}{\mathrm{f}(x_1)}\;\underset{r_2-1-k_2-m_1-r_1}{\mathrm{F}(x_2-)-\mathrm{F}(x_1)}\;\underset{k_2+1+m_2}{\mathrm{f}(x_2)}\;\dots\;\underset{k_p+1+m_p}{\mathrm{f}(x_p)}\;\underset{n-m_p-r_p}{1-\mathrm{F}(x_p)}]\,, \tag{3}$$

where $\mathrm{C}=\prod_{w=1}^{p+1}[(r_w-1-k_w-m_{w-1}-r_{w-1})!]^{-1}\cdot\prod_{w=1}^{p}[(k_w+1+m_w)!]^{-1}$, $m_{w-1}+k_w\le r_w-r_{w-1}-1$, $m_0=0$, $k_{p+1}=0$, $\mathrm{F}(x_w)=(F_1(x_w),F_2(x_w),\dots,F_n(x_w))'$, $\mathrm{f}(x_w)=(f_1(x_w),f_2(x_w),\dots,f_n(x_w))'$ and $F_i(x_w-)=P(X_i<x_w)$.

(3) can be written as

$$f_{r_1,r_2,\dots r_p:n}(x_1,x_2,\dots,x_p)$$

$$=\sum_{m_p,k_p\dots,m_1,k_1}\mathrm{C}\sum_{n_{s_1},n_{s_2},\dots,n_{s_{4p}}}\left(\prod_{w=1}^{p+1}per[\underset{m_{w-1}}{\mathrm{f}(x_{w-1})}][s_{4(w-1)}/.)\,per[\underset{r_w-1-k_w-m_{w-1}-r_{w-1}}{\mathrm{F}(x_w-)-\mathrm{F}(x_{w-1})}][s_{4w-3}/.)\,per[\underset{k_w}{\mathrm{f}(x_w)}][s_{4w-2}/.)\right)$$

$$\cdot\prod_{w=1}^{p}per[\underset{1}{\mathrm{f}(x_w)}][s_{4w-1}/.), \tag{4}$$

where $\sum_{n_{s_1},n_{s_2},\ldots,n_{s_{4p}}}$ denotes the sum over $\bigcup_{l=1}^{4p} s_l$ for which $s_\upsilon \cap s_\vartheta = \phi$ for $\upsilon \neq \vartheta$

$$\bigcup_{l=1}^{4p+1} s_l = \{1,2,\ldots,n\}, \quad s_l = \begin{cases} \{s_l^{(1)}, s_l^{(2)}, \ldots, s_l^{(m_{\frac{l}{4}})}\}, & \text{if } l \equiv 0 \pmod 4 \\ \{s_l^{(1)}, s_l^{(2)}, \ldots, s_l^{(r_{\frac{l+3}{4}}-1-m_{\frac{l-1}{4}}-k_{\frac{l+3}{4}}-r_{\frac{l-1}{4}})}\}, & \text{if } l \equiv 1 \pmod 4 \\ \{s_l^{(1)}, s_l^{(2)}, \ldots, s_l^{(k_{\frac{l+2}{4}})}\}, & \text{if } l \equiv 2 \pmod 4 \\ \{s_l^{(1)}\}, & \text{if } l \equiv 3 \pmod 4. \end{cases}$$

(4) can be expressed as

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)$$

$$= \sum_{m_p,k_p\ldots,m_1,k_1} C \sum_{n_{s_1},n_{s_2},\ldots,n_{s_{4p}}} \left(\prod_{w=1}^{p+1} per[\underset{m_{w-1}}{\mathrm{f}(x_{w-1})}][s_{4(w-1)}/.)\, per[\underset{r_w-1-k_w-m_{w-1}-r_{w-1}}{\mathrm{F}(x_w-)-\mathrm{F}(x_{w-1})}][s_{4w-3}/.)\, per[\underset{k_w}{\mathrm{f}(x_w)}][s_{4w-2}/.) \right)$$

$$\cdot \left(\prod_{w=1}^{p} per[\underset{1}{\mathrm{f}(x_w)}][s_{4w-1}/.) \right) \int_0^1\int_0^1 \cdots \int_0^1 \prod_{w=1}^{p} \frac{(k_w+1+m_w)!}{k_w!\, m_w!} y_w^{k_w}(1-y_w)^{m_w}\, dy_w. \quad (5)$$

The identity (5) can also be written as

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = \sum_{m_p,k_p\ldots,m_1,k_1} \left(\prod_{w=1}^{p+1} \frac{1}{(r_w-1-k_w-m_{w-1}-r_{w-1})!\, m_{w-1}!\, k_w!} \sum_{n_{s_1},n_{s_2},\ldots,n_{s_{4p}}} \int_0^1\int_0^1 \cdots \int_0^1 \right.$$

$$\left. \cdot per[\underset{m_{w-1}}{(1-y_{w-1})\mathrm{f}(x_{w-1})}][s_{4(w-1)}/.)\, per[\underset{r_w-1-k_w-m_{w-1}-r_{w-1}}{\mathrm{F}(x_w-)-\mathrm{F}(x_{w-1})}][s_{4w-3}/.)\, per[\underset{k_w}{y_w\mathrm{f}(x_w)}][s_{4w-2}/.) \right) \left(\prod_{w=1}^{p} per[\underset{1}{dy_w\mathrm{f}(x_w)}][s_{4w-1}/.) \right).$$

(6)

In (6), if $\mathrm{v}^{(w)} = y_w \mathrm{f}(x_w) + \mathrm{F}(x_w-)$, the following identity is obtained.

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = \sum_{m_p,k_p\ldots,m_1,k_1} \left(\prod_{w=1}^{p+1} \frac{1}{(r_w-1-k_w-m_{w-1}-r_{w-1})!\, m_{w-1}!\, k_w!} \sum_{n_{s_1},n_{s_2},\ldots,n_{s_{4p}}} \int_{F_{s_3}(x_1-)}^{F_{s_3}(x_1)} \int_{F_{s_7}(x_2-)}^{F_{s_7}(x_2)} \cdots \int_{F_{s_{4p-1}}(x_p-)}^{F_{s_{4p-1}}(x_p)} \right.$$

$$\left. \cdot per[\underset{m_{w-1}}{\mathrm{F}(x_{w-1})-\mathrm{v}^{(w-1)}}][s_{4(w-1)}/.)\, per[\underset{r_w-1-k_w-m_{w-1}-r_{w-1}}{\mathrm{F}(x_w-)-\mathrm{F}(x_{w-1})}][s_{4w-3}/.)\, per[\underset{k_w}{\mathrm{v}^{(w)}-\mathrm{F}(x_w-)}][s_{4w-2}/.) \right) \prod_{w=1}^{p} per[\underset{1}{d\mathrm{v}^{(w)}}][s_{4w-1}/.).$$

(7)

By considering

$$\sum_{k_w=0}^{\ell} \sum_{m_{w-1}=0}^{\ell} \frac{1}{k_w!\, m_{w-1}!\, (\ell-k_w-m_{w-1})!} \sum_{n_{s_{4(w-1)}},n_{s_{4w-3}},n_{s_{4w-2}}}$$

$$\cdot\, per[\underset{m_{w-1}}{\mathrm{x}^{(1)}}][s_{4(w-1)}/.)\, per[\underset{\ell-k_w-m_{w-1}}{\mathrm{x}^{(2)}}][s_{4w-3}/.)\, per[\underset{k_w}{\mathrm{x}^{(3)}}][s_{4w-2}/.)\, per[\underset{1}{\mathrm{x}^{(4)}}][s_{4w-1}/.)$$

$$=\frac{1}{\ell!}\sum_{n_{\varsigma_{2w-1}}} per[\mathrm{x}^{(1)}+\underset{\ell}{\mathrm{x}^{(2)}}+\mathrm{x}^{(3)}][\varsigma_{2w-1}/.)per[\underset{1}{\mathrm{x}^{(4)}}][\varsigma_{2w}/.)\,, \qquad (8)$$

where $m_{w-1}+k_w \le \ell$ and using (8) in (7), we get

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\mathrm{D}\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}}\int\left(\prod_{w=1}^{p+1}per[\mathrm{F}(x_{w-1})-\mathrm{v}^{(w-1)}+\underset{r_w-r_{w-1}-1}{\mathrm{F}(x_w-)-\mathrm{F}(x_{w-1})+\mathrm{v}^{(w)}-\mathrm{F}(x_w-)}][\varsigma_{2w-1}/.)\right)$$

$$\cdot\prod_{w=1}^{p}per[\underset{1}{\mathrm{dv}^{(w)}}][\varsigma_{2w}/.),$$

where $\varsigma_{2w-1}=s_{4(w-1)}\cup s_{4w-3}\cup s_{4w-2}$ and $\varsigma_{2w}=s_{4w-1}$. Thus, the proof is completed.

If $x_1=x_2=\ldots=x_p=x$, it should be written $\iint\ldots\int$ instead of $\int$ in (1), where $\iint\ldots\int$ is to be carried out over the region: $F_{\varsigma_2^{(1)}}(x_1-)\le v^{(1)}_{\varsigma_2^{(1)}}\le v^{(2)}_{\varsigma_4^{(1)}}\le\ldots\le v^{(p)}_{\varsigma_{2p}^{(1)}}\le F_{\varsigma_{2p}^{(1)}}(x_p)$, $F_{\varsigma_2^{(1)}}(x_1-)\le v^{(1)}_{\varsigma_2^{(1)}}\le F_{\varsigma_2^{(1)}}(x_1)$, $F_{\varsigma_4^{(1)}}(x_2-)\le v^{(2)}_{\varsigma_4^{(1)}}\le F_{\varsigma_4^{(1)}}(x_2)$, . . . , $F_{\varsigma_{2p}^{(1)}}(x_p-)\le v^{(p)}_{\varsigma_{2p}^{(1)}}\le F_{\varsigma_{2p}^{(1)}}(x_p)$.

Moreover, if $x_1\le x_2\le\ldots\le x_p$, it should be written $\iint\ldots\int$ instead of $\int$ in (1), where $\iint\ldots\int$ is to be carried out over the region: $v^{(1)}_{\varsigma_2^{(1)}}\le v^{(2)}_{\varsigma_4^{(1)}}\le\ldots\le v^{(p)}_{\varsigma_{2p}^{(1)}}$, $F_{\varsigma_2^{(1)}}(x_1-)\le v^{(1)}_{\varsigma_2^{(1)}}\le F_{\varsigma_2^{(1)}}(x_1)$, $F_{\varsigma_4^{(1)}}(x_2-)\le v^{(2)}_{\varsigma_4^{(1)}}\le F_{\varsigma_4^{(1)}}(x_2)$, . . . , $F_{\varsigma_{2p}^{(1)}}(x_p-)\le v^{(p)}_{\varsigma_{2p}^{(1)}}\le F_{\varsigma_{2p}^{(1)}}(x_p)$.

We will now express the following theorem to obtain the joint *df* of *p* order statistics of *innid* discrete random variables.

Theorem 3.2.

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\mathrm{D}\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}}\int_{\mathrm{V}}\left(\prod_{w=1}^{p+1}per[\underset{r_w-r_{w-1}-1}{\mathrm{v}^{(w)}-\mathrm{v}^{(w-1)}}][\varsigma_{2w-1}/.)\right)\prod_{w=1}^{p}per[\underset{1}{\mathrm{dv}^{(w)}}][\varsigma_{2w}/.)\,. \qquad (9)$$

Proof. It can be written

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{z_p,\ldots,z_2,z_1}f_{r_1,r_2,\ldots,r_p:n}(z_1,z_2,\ldots,z_p)\,.$$

Using (1) in the above identity, we have

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{z_p,\ldots,z_2,z_1}\mathrm{D}\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}}\int\left(\prod_{w=1}^{p+1}per[\underset{r_w-r_{w-1}-1}{\mathrm{v}^{(w)}-\mathrm{v}^{(w-1)}}][\varsigma_{2w-1}/.)\right)\prod_{w=1}^{p}per[\underset{1}{\mathrm{dv}^{(w)}}][\varsigma_{2w}/.).$$

Thus, the proof is completed.

4. RESULTS FOR DISTRIBUTION AND PROBABILITY FUNCTIONS

In this section, the results related to *pf* and *df* of $X_{r_1:n},X_{r_2:n},\ldots,X_{r_p:n}$ will be given. We will express the following result for *pf* of the *r*th order statistic of *innid* discrete random variables.

Result 4.1.

$$f_{r_1:n}(x_1)=\frac{1}{(r_1-1)!(n-r_1)!}\sum_{n_{\varsigma_1},n_{\varsigma_2}}\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)} per[\underset{r_1-1}{\mathrm{v}^{(1)}}][\varsigma_1/.)\,per[\underset{1}{\mathrm{dv}^{(1)}}][\varsigma_2/.)\,per[\underset{n-r_1}{1-\mathrm{v}^{(1)}}][\varsigma_3/.)\cdot \quad 10)$$

Proof. In (1), if $p=1$, (10) is obtained.

Specially, in (10), by taking $n=2$, $p=1$ and $r_1=2$, the following identity is obtained where $\mathrm{v}^{(1)}=(v_1^{(1)},v_2^{(1)})'$ and $\mathrm{dv}^{(1)}=(dv_1^{(1)},dv_2^{(1)})'$.

$$f_{2:2}(x_1)=\sum_{n_{\varsigma_1}=1}\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)} v_{\varsigma_1^{(1)}}^{(1)}dv_{\varsigma_2^{(1)}}^{(1)}$$

$$f_{2:2}(x_1)=\int_{F_2(x_1-)}^{F_2(x_1)} v_1^{(1)}dv_2^{(1)}+\int_{F_1(x_1-)}^{F_1(x_1)} v_2^{(1)}dv_1^{(1)}$$

$$\left(v_1^{(1)}=[v_2^{(1)}-F_2(x_1-)]\frac{f_1(x_1)}{f_2(x_1)}+F_1(x_1-),\ v_2^{(1)}=[v_1^{(1)}-F_1(x_1-)]\frac{f_2(x_1)}{f_1(x_1)}+F_2(x_1-)\right)$$

$$=[[\frac{\left(v_2^{(1)}\right)^2}{2}-F_2(x_1-)v_2^{(1)}]\frac{f_1(x_1)}{f_2(x_1)}+F_1(x_1-)v_2^{(1)}]\Bigg|_{F_2(x_1-)}^{F_2(x_1)}+[[\frac{\left(v_1^{(1)}\right)^2}{2}-F_1(x_1-)v_1^{(1)}]\frac{f_2(x_1)}{f_1(x_1)}+F_2(x_1-)v_1^{(1)}]\Bigg|_{F_1(x_1-)}^{F_1(x_1)}$$

$$=\frac{[F_2(x_1)+F_2(x_1-)]}{2}f_1(x_1)-F_2(x_1-)f_1(x_1)+F_1(x_1-)f_2(x_1)]+\frac{[F_1(x_1)+F_1(x_1-)]}{2}f_2(x_1)-F_1(x_1-)f_2(x_1)+F_2(x_1-)f_1(x_1)]$$

$$=\frac{[2F_2(x_1)-f_2(x_1)]}{2}f_1(x_1)-[F_2(x_1)-f_2(x_1)]f_1(x_1)+[F_1(x_1)-f_1(x_1)]f_2(x_1)]+$$

$$+\frac{[2F_1(x_1)-f_1(x_1)]}{2}f_2(x_1)-[F_1(x_1)-f_1(x_1)]f_2(x_1)+[F_2(x_1)-f_2(x_1)]f_1(x_1)]$$

$$=F_1(x_1)f_2(x_1)+F_2(x_1)f_1(x_1)-f_1(x_1)f_2(x_1).$$

In Result 4.2 and Result 4.3, the *pf* of minimum and maximum order statistics of *innid* discrete random variables are given, respectively.

Result 4.2.

$$f_{1:n}(x_1)=\frac{1}{(n-1)!}\sum_{n_{\varsigma_2}}\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)} per[\underset{1}{\mathrm{dv}^{(1)}}][\varsigma_2/.)\,per[\underset{n-1}{1-\mathrm{v}^{(1)}}][\varsigma_3/.)\cdot \quad (11)$$

Proof. In (10), if $r_1=1$, (11) is obtained.

Result 4.3.

$$f_{n:n}(x_1)=\frac{1}{(n-1)!}\sum_{n_{\varsigma_1}}\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)} per[\underset{n-1}{\mathrm{v}^{(1)}}][\varsigma_1/.)\,per[\underset{1}{\mathrm{dv}^{(1)}}][\varsigma_2/.)\cdot \quad (12)$$

Proof. In (10), if $r_1=n$, (12) is obtained.

In the following result, we will give the joint *pf* of $X_{1:n},X_{2:n},\ldots,X_{p:n}$.

Result 4.4. If $x_1\le x_2\le\ldots\le x_p$,

$$f_{1,2,\ldots,p:n}(x_1,x_2,\ldots,x_p)=\frac{1}{(n-p)!}\sum_{n_{\varsigma_2},n_{\varsigma_4},\ldots,n_{\varsigma_{2p}}}\iint\ldots\int per[\underset{n-p}{1-\mathrm{v}^{(p)}}][\varsigma_{2p+1}/.)\prod_{w=1}^{p}per[\underset{1}{\mathrm{dv}^{(w)}}][\varsigma_{2w}/.), \quad (13)$$

where $\iint\ldots\int$ is to be carried out over the region: $v^{(1)}_{\varsigma^{(1)}_2} \le v^{(2)}_{\varsigma^{(1)}_4} \le \ldots \le v^{(p)}_{\varsigma^{(1)}_{2p}}$, $F_{\varsigma^{(1)}_2}(x_1-) \le v^{(1)}_{\varsigma^{(1)}_2} \le F_{\varsigma^{(1)}_2}(x_1)$, $F_{\varsigma^{(1)}_4}(x_2-) \le v^{(2)}_{\varsigma^{(1)}_4} \le F_{\varsigma^{(1)}_4}(x_2)$, ..., $F_{\varsigma^{(1)}_{2p}}(x_p-) \le v^{(p)}_{\varsigma^{(1)}_{2p}} \le F_{\varsigma^{(1)}_{2p}}(x_p)$.

Proof. In (1), if $r_1 = 1$, $r_2 = 2, \ldots, r_p = p$ and $\iint\ldots\int$ instead of $\int$, (13) is obtained. We will now give three results for the *df* of single order statistic of *innid* discrete random variables.

Result 4.5.

$$F_{r_1:n}(x_1) = \frac{1}{(r_1-1)!(n-r_1)!}\sum_{n_{\varsigma_1},n_{\varsigma_2}}\int_0^{F_{\varsigma^{(1)}_2}(x_1)} per[\underset{r_1-1}{\mathrm{v}^{(1)}}][\varsigma_1/.)\, per[\underset{1}{\mathrm{dv}^{(1)}}][\varsigma_2/.)\, per[\underset{n-r_1}{1-\mathrm{v}^{(1)}}][\varsigma_3/.)\ . \qquad (14)$$

Proof. In (9), if $p = 1$, (14) is obtained.

Result 4.6.

$$F_{1:n}(x_1) = \frac{1}{(n-1)!}\sum_{n_{\varsigma_2}}\int_0^{F_{\varsigma^{(1)}_2}(x_1)} per[\underset{1}{\mathrm{dv}^{(1)}}][\varsigma_2/.)\, per[\underset{n-1}{1-\mathrm{v}^{(1)}}][\varsigma_3/.)\ . \qquad (15)$$

Proof. In (14), if $r_1 = 1$, (15) is obtained.

Result 4.7.

$$F_{n:n}(x_1) = \frac{1}{(n-1)!}\sum_{n_{\varsigma_1}}\int_0^{F_{\varsigma^{(1)}_2}(x_1)} per[\underset{n-1}{\mathrm{v}^{(1)}}][\varsigma_1/.)\, per[\underset{1}{\mathrm{dv}^{(1)}}][\varsigma_2/.)\ . \qquad (16)$$

Proof. In (14), if $r_1 = n$, (16) is obtained.

In the following result, we will give the joint *df* of $X_{1:n}, X_{2:n}, \ldots, X_{p:n}$.

Result 4.8.

$$F_{1,2,\ldots,p:n}(x_1,x_2,\ldots,x_p) = \frac{1}{(n-p)!}\sum_{n_{\varsigma_2},n_{\varsigma_4},\ldots,n_{\varsigma_{2p}}}\int_{\mathrm{V}} per[\underset{n-p}{1-\mathrm{v}^{(p)}}][\varsigma_{2p+1}/.)\prod_{w=1}^{p} per[\underset{1}{\mathrm{dv}^{(w)}}][\varsigma_{2w}/.)\ . \qquad (17)$$

Proof. In (9), if $r_1 = 1$, $r_2 = 2, \ldots, r_p = p$, (17) is obtained.

REFERENCES

[1] Arnold, B.C., Balakrishnan, N., and Nagaraja H.N., (1992). A First Course in Order Statistics, John Wiley and Sons Inc., New York.
[2] Balakrishnan, N., (1986). Order Statistics from Discrete Distributions, Commun. Statist. Theor. Meth. 15, No.3, 657-675.
[3] Balakrishnan, N., (2007). Permanents, Order Statistics, Outliers and Robustness, Rev. Mat. Complut. 20, No:1, 7-107.
[4] Balasubramanian, K. and Beg, M.I., (2003). On Special Linear Identities for Order Statistics, Statistics 37, No:4, 335-339.
[5] Balasubramanian, K. Beg, M.I., And Bapat, R.B., (1991). On Families of Distributions Closed Under Extrema, Sankhyā Ser. A 53, No:3, 375-388.
[6] Balasubramanian, K., Beg M.I., and Bapat, R.B., (1996). An Identity for The Joint Distribution of Order Statistics and Its Applications, J. Statist. Plann. Inference 55, No.1, 13-21, 1996.

[7] Balasubramanian, K., Balakrishnan, N., and Malik, H.J., (1994). Identities for Order Statistics from Non-Independent Non-Identical Variables, Sankhyā Ser. B 56, No:1, 67-75.
[8] Bapat, R.B. and Beg, M.I., (1989). Order Statistics for Nonidentically Distributed Variables and Permanents, Sankhyā Ser. A 51, No:1, 79-93.
[9] Beg, M.I., (1991). Recurrence Relations and Identities for Product Moments of Order Statistics Corresponding to Nonidentically Distributed Variables, Sankhyā Ser. A 53, No.3, 365-374.
[10] Cao, G. and West, M., (1997). Computing Distributions of Order Statistics, Communications in Statistics Theory and Methods 26, No.3, 755-764.
[11] Childs, A. and Balakrishnan, N., (2006). Relations for Order Statistics from Non-Identical Logistic Random Variables and Assessment of The Effect of Multiple Outliers on Bias of Linear Estimators, Journal of Statistical Planning And Inference 136, No.7, 2227-2253.
[12] Corley, H.W., (1984). Multivariate Order Statistics, Commun. Statist. Theor. Meth. 13, No.10, 1299-1304.
[13] Cramer, E., Herle K., and Balakrishnan, N., (2009). Permanent Expansions and Distributions of Order Statistics in The Inid Case, Communications in Statistics Theory and Methods 38, No:12, 2078-2088.
[14] David, H.A., (1981). Order Statistics, John Wiley and Sons Inc., New York.
[15] Gan, G. and Bain, L.J., (1995). Distribution of Order Statistics for Discrete Parents with Applications to Censored Sampling, J. Statist. Plann. Inference 44, No:1, 37-46, 1995.
[16] Goldie, C.M. and Maller, R.A., (1999). Generalized Densities of Order Statistics, Statistica Neerlandica 53, No:2, 222-246, 1999.
[17] Guilbaud, O., (1982). Functions of Non-I.I.D. Random Vectors Expressed as Functions Of I.I.D. Random Vectors, Scand. J. Statist. 9, No.4, 229-233.
[18] Khatri, C.G., (1962). Distributions of Order Statistics for Discrete Case, Annals of The Ins. of Stat. Math. 14, No:1, 167-171.
[19] Nagaraja, H.N., (1986). Structure of Discrete Order Statistics, J. Statist. Plann. Inference 13, No.1, 165-177, 1986.
[20] Nagaraja, H.N., (1992). Order Statistics from Discrete Distributions, Statistics 23, No.3, 189-216, 1992.
[21] Reiss, R.D., (1989). Approximate Distributions of Order Statistics, Springer-Verlag, New York.
[22] Vaughan, R.J., and Venables, W.N., (1972). Permanent Expressions for Order Statistics Densities, Journal of The Royal Statistical Society Ser. B 34, No.2, 308-310, 1972.