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# SOME INEQUALITIES FOR POLYNOMIAL FUNCTIONS

**ABSRACT**

In this work, we first show that the inequality, established in the unit disc for maximum modulus of polynomial functions [12], also holds for any disc of radius $R<\infty$. In the case where polynomials have $z=0$ as a multiple root, and also for the univalent polynomial functions $f:C\to C$ with $f(0)=0$, $f(a)=a^q$, $f(-a)=-a^q$, we obtain different forms of this inequality. Then we attain quite distinct new inequalities for univalent polynomial functions in both the unit disc and disc of an arbitrary radius $R<\infty$.

**Keywords:** Mathematicle Analysis, Polynomial Functions, Univalent Function With Three Preassigned Values, Maximum Modulus Values, Inequalities.

## POLİNOM FONKSİYONLAR İÇİN BAZI EŞİTSİZLİKLER

**ÖZET**

Bu çalışmada, önce [12] de birim diskte polinomların maksimum modülleri için ispatlanan eşitsizliğin, herhangi bir $R<\infty$ yarıçaplı disk için de geçerli olduğu gösterildi. $z=0$ noktası polinomların katlı kökü olması durumunda ve $f(0)=0$, $f(a)=a^q$, $f(-a)=-a^q$ olan $f:C\to C$ ünivalent polinomlar için, bu eşitsizliğin farklı formları elde edilmiştir. Ayrıca ,ünivalent polinom fonksiyonlar için bu eşitsizliklerden tamamen farklı yeni eşitsizlikler hem birim diskte hem de herhangi bir $R<\infty$ yarıçaplı bir diskte elde edilmiştir.

**Anahtar Kelimeler:** Matematiksel Analiz, Polinom Fonksiyonlar, Ön Görülen Üç Değeri Alan Ünivalent Fonksiyonlar, Maksimum Modül Değerler, Eşitsizlikler

**1. INTRODUCTION (GİRİŞ)**

Let $f: C \to C$ be a polynomial function with complex variable and let $M_f = \max_{|z|=1} |f(z)|$. Ostrowski [12], Rassias [13], Mohr [11] and Çelik [6] investigated inequalities in the unit disc between $M_{f_1 \cdot f_2 \cdots f_n}$ and $M_{f_1}.M_{f_2}...M_{f_n}$ ($n \geq 2$) for complex polynomial functions. The same problem is investigated in the disc of radius $R > 1$ and also on two hyperbolic region by Çelik [3 and 5], respectively. Rassias [13] proved the following theorem:

**Theorem R.** Let be $f_1, f_2, ..., f_n : C \to C$ ($n \geq 2$) complex-valued polynomial functions of degrees $d_1, d_2, ..., d_n$ respectively, of a complex variable $z$.

Define $M_f = \max_{|z|=1} |f(z)|$. Then

$$M_{f_1}.M_{f_2}...M_{f_n} \geq M_{f_1 \cdot f_2 \cdots f_n} \geq k.\, M_{f_1}.M_{f_2}...M_{f_n} \quad (1)$$

where

$$k = (Sin \frac{2}{n} \frac{\pi}{8d_1})^{d_1} . (Sin \frac{2}{n} \frac{\pi}{8d_2})^{d_2} ... (Sin \frac{2}{n} \frac{\pi}{8d_n})^{d_n}$$

A function that is one to one and analytic in a region $A \subset C$ is called a univalent function in $A$ [2, 4 and 8].

The set of univalent functions with $f(0) = 0$, $f(a) = a$, $f(-a) = -a$ is denoted by $T(a)$ in Avcı and Zlotkiewicz [2] , and they called attention to the class, $f(z) = z + \rho(a^2 z^2 - z^4)$, $(2a^2 + 4)\rho \leq 1$.

Now let the region $A \subset C$ be the unit disc or any disc with radius $R < \infty$. Let $\overline{A} = A \cup \partial A$ where $\partial A$ stands for the boundary of $A$. Let us denote by $UTP(a)$ the set of univalent polynomial functions with $f: C \to C$

$f(0) = 0$, $f(a) = a^q$, $f(-a) = -a^q$, where $a, -a \in \overline{A}$ and $q \in N$.

In $UTP(a)$, we have the following set for polynomial functions of degre $4m$ and $q = 1,3,5,...,4m-1$:

The set of those functions of the form

$f(z) = z^q + \sum_{k=1}^{m} \rho_k (z^{4k} - a^{2k} z^{2k})$ $(\rho_k \in C, \rho_m = 1)$ is denoted by $UT_{2,2}^4 P(a)$.

And, similar sets are denoted by $UT_{3,1}^4 P(a)$, $UT_{1,3}^4 P(a)$, $UT_{4,0}^4 P(a)$, respectively.

Throughout our work we use the usual topology on $C$. Also 1) if $A$ the unit disc, we take $M_f = \max_{|z|=1} |f(z)|$. 2) if $A$ is any disc with radius $R < \infty$ we define $M'_f = \max_{|z|=R} |f(z)|$. In our work we will deal with the set $UT_{2,2}^4 P(a^q)$.

## 2. RESEARCH SIGNIFICANSE (ÇALIŞMANIN ÖNEMİ)

Let $f, g : C \to C$ be two polynomial fonctions with a complex variable $z$. In the unit disc,we define $M_f = \max_{|z|=1} |f(z)|$ and $M_g = \max_{|z|=1} |g(z)|$. In [6, 11, 12, and 13] inequalities between $M_{f.g}$ and $M_f . M_g$ are showen. Here, fixed points and their powers were not considered in any of the coefficients of the inequalities. In this work, we show that the inequality, etablished in the unit disc [12], also holds for any disc of radius $R$. Also,let A be a disc of a arbitrary radius $R < \infty$, for $UT^4_{2,2}P(a)$ we define $M'_f = \max_{|z|=R} |f(z)|$ and $M'_g = \max_{|z|=R} |g(z)|$. Then we attain new inequalities between $M'_{f.g}$ and $M'_f . M'_g$.

In those inequalities fixed points,and their powers,of the polynomials
functions are considered.

## 3. ANALYTICAL STUDY (ANALİTİK ÇALIŞMA)

Our work is baded on pure mathematics. Therefore, we deduce relations (fomulas) and equations (analytical relations) by means of theoritical methods, which are prof techniques. As usual, these methods are carried in terms of hypotheses-conclusions.

## 4. MAIN RESULTS ON THEOREM R (THEOREM R İÇİN SONUÇLAR)

In this section we derive some consequences of Theorem R.
In Rassias [12], for a polynomial function $f : C \to C$ of degree $d$ written in the form $f(z) = (z - z_1).(z - z_2)...(z - z_d)$ and letting $M_f = \max_{|z|=1} |f(z)|$, it is proved that " for each $\varepsilon > 0$ and $n > 0$, we have for most $\theta$ that

$$\left| f(e^{i\theta}) \right| = \left| e^{i\theta} - z_1 \right| . \left| e^{i\theta} - z_2 \right| ... \left| e^{i\theta} - z_d \right| \geq M_f . \left( \sin \frac{2}{n} \frac{\pi}{8d + \varepsilon} \right)^d$$

or

$$\frac{\left| e^{i\theta} - z_1 \right| . \left| e^{i\theta} - z_2 \right| ... \left| e^{i\theta} - z_d \right|}{\left| 1 + |z_1| \right| . \left| 1 + |z_2| \right| ... \left| 1 + |z_d| \right|} \geq \left[ \sin \frac{2}{n} \frac{\pi}{8d + \varepsilon} \right]^d \text{."}$$

**Lemma 1.** Let each $\varepsilon > 0$ and $n > 0$, for most $\theta$, $0 < R < \infty$ and $|a_k| \leq R \quad (k = 1, 2, .., d)$. Then

$$\frac{\left| \mathrm{R}e^{i\theta} - a_1 \right| . \left| \mathrm{R}e^{i\theta} - a_2 \right| ... \left| \mathrm{R}e^{i\theta} - a_d \right|}{\left| R + |a_1| \right| . \left| R + |a_2| \right| ... \left| R + |a_d| \right|} \geq \left[ \sin \frac{2}{n} \frac{\pi}{8d + \varepsilon} \right]^d \tag{2}$$

holds if and only if

$$\frac{|e^{i\theta}-z_1|.|e^{i\theta}-z_2|...|e^{i\theta}-z_d|}{|1+|z_1||.|1+|z_2||...|1+|z_d||} \geq \left[\sin\frac{2}{n}\frac{\pi}{8d+\varepsilon}\right]^d \qquad (3)$$

where $|z_k| \leq 1 \quad (k=1,2,..,d)$.

**Proof.** Assume that (2) holds. Then we have

$$\frac{\left|e^{i\theta}-\frac{a_1}{R}\right|.\left|e^{i\theta}-\frac{a_2}{R}\right|...\left|e^{i\theta}-\frac{a_d}{R}\right|}{\left(1+\frac{|a_1|}{R}\right).\left(1+\frac{|a_2|}{R}\right)...\left(1+\frac{|a_d|}{R}\right)} \geq \left[\sin\frac{2}{n}\frac{\pi}{8d+\varepsilon}\right]^d \qquad (4)$$

Let us take $\frac{a_k}{R} = z_k \ (k=1,2,..,d)$ in (4); this yields (3) since $|z_k| \leq 1 \quad (k=1,2,..,d)$. Conversely, assume that (3) is true. Then

$$\frac{|\mathrm{Re}^{i\theta}-Rz_1|.|\mathrm{Re}^{i\theta}-Rz_2|...|\mathrm{Re}^{i\theta}-Rz_d|}{|R+R|z_1||.|R+R|z_2||...|R+R|z_d||} \geq \left[\sin\frac{2}{n}\frac{\pi}{8d+\varepsilon}\right]^d \qquad (5)$$

holds, where $0<R<\infty$. Take $R.z_k = a_k \ (k=1,2,..,d)$ in (5); this implies (2) since $|a_k| \leq R \quad (k=1,2,..,d)$.

From now on $A$ will be any disc with radius $R<\infty$, $n \geq 2$, and polynomial functions $f_1, f_2,..., f_n : C \rightarrow C$ will be considered.

**Theorem 1.** Let $d_1, d_2,..., d_n$ be the degrees of polynomial functions $f_1, f_2,..., f_n$ respectively. Then

$$M'_{f_1 \cdot f_2 \cdots f_n} \geq \dot{k} \ . \ M'_{f_1}.M'_{f_2}...M'_{f_n} \qquad (6)$$

where

$$\dot{k} = (Sin\frac{2}{n}\frac{\pi}{8d_1})^{d_1} . (Sin\frac{2}{n}\frac{\pi}{8d_2})^{d_2} ... (Sin\frac{2}{n}\frac{\pi}{8d_n})^{d_n} \text{ (or } k=\dot{k}\text{)}.$$

**Proof.** It suffices to consider Lemma 1 and Theorem R.

**Lemma 2.** Let each $\varepsilon > 0$ and $n > 0$, for most $\theta$, $0<R<\infty$ and $|a_k| \leq R$ $(k=1,2,..,d-r)$.

Then

$$\frac{|\mathrm{Re}^{i\theta}-a_1|.|\mathrm{Re}^{i\theta}-a_2|...|\mathrm{Re}^{i\theta}-a_{d-r}|}{|R+|a_1||.|R+|a_2||...|R+|a_{d-r}||} \geq \left[\sin\frac{2}{n}\frac{\pi}{8(d-r)+\varepsilon}\right]^{d-r}$$

holds if and only if

$$\frac{|e^{i\theta}-z_1|.|e^{i\theta}-z_2|...|e^{i\theta}-z_{d-r}|}{|1+|z_1||.|1+|z_2||...|1+|z_{d-r}||} \geq \left[\sin\frac{2}{n}\frac{\pi}{8(d-r)+\varepsilon}\right]^{d-r}$$

where

$|z_k| \le 1 \quad (k = 1,2,..,d-r)$ .

**Proof.** It suffices to replace $d$ by $d-r$ in Lemma 1.

**Theorem 2.**Let $d_1, d_2, ..., d_n$ be the degrees of polynomial functions $f_1, f_2, ..., f_n$, respectively, which have the zero point as the multiple roots $r_1, r_2, ..., r_n$. Then

$$M'_{f_1 \cdot f_2 \cdots f_n} \ge k_1 \, . \, M'_{f_1} . M'_{f_2} ... M'_{f_n} \tag{7}$$

where

$$k_1 = (Sin \frac{2}{n} \frac{\pi}{8(d_1 - r_1})^{d_1 - r_1} \, . \, (Sin \frac{2}{n} \frac{\pi}{8(d_2 - r_2)})^{d_2 - r_2} ... (Sin \frac{2}{n} \frac{\pi}{8(d_n - r_n})^{d_n - r_n}$$

**Proof.** Given that $R < \infty$ and $|a_k| \le R \quad (k = 1,2,..,d-r)$, by hypothesis, each polynomial function $f : C \to C$ can be written in this from

$f_j(z) = r^j . (z - a_1).(z - a_2)...(z - a_{d_j - r_j})$ . For these functions we have

$$M'_f = \underset{|z|=R}{Max} |f_j(z)| \le R^{r_j} . (R + |a_1|).(R + |a_2|)...(R + |a_{d_j - r_j}|)$$

and

$$|f_j(\mathrm{Re}^{i\theta})| = R^{r_j} . |\mathrm{Re}^{i\theta} - a_1| . |\mathrm{Re}^{i\theta} - a_2| ... |\mathrm{Re}^{i\theta} - a_{d_j - r_j}| \, .$$

From these two expessions we can write

$$\frac{|f_j(\mathrm{Re}^{i\theta})|}{M'_{f_j}} \ge \frac{|\mathrm{Re}^{i\theta} - a_1| . |\mathrm{Re}^{i\theta} - a_2| ... |\mathrm{Re}^{i\theta} - a_{d_j - r_j}|}{|R + |a_1|| . |R + |a_2|| ... |R + |a_{d_j - r_j}||} \qquad (j = 1,2...,n) \, .$$

Then we apply Lemma 2 and Theorem R.

**Corollary 1.**Let $f_1, f_2, ..., f_j$, $j \le n$ be polynomial functions of degrees $d_1, d_2, ..., d_j$, respectively, which accept the zero point as multiple roots $r_1, r_2, ..., r_j$, respectively; let $f_{j+1}, f_{j+2}, ..., f_n$, in number $n - j$, be polynomial functions of degrees $d_{j+1}, d_{j+2}, ..., d_n$ which don't have the zero point as multiple roots. Then

$$M'_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \ge k_2 \, . \, M'_{f_1} . M'_{f_2} ... M'_{f_j} . M'_{f_{j+1}} ... M'_{f_n} \tag{8}$$

where

$$k_1 = (Sin \frac{2}{n} \frac{\pi}{8(d_1 - r_1})^{d_1 - r_1} \, . \, (Sin \frac{2}{n} \frac{\pi}{8(d_2 - r_2)})^{d_2 - r_2} ... (Sin \frac{2}{n} \frac{\pi}{8(d_j - r_j})^{d_j - r_j}$$

$$. \, (Sin \frac{2}{n} \frac{\pi}{8d_{j+1}})^{d_{j+1}} \, . \, (Sin \frac{2}{n} \frac{\pi}{8d_{j+2}})^{d_{j+2}} ... (Sin \frac{2}{n} \frac{\pi}{8d_n})^{d_n}$$

**Proof.** Putting $F(z) = f_1(z).f_2(z)...f_n(z) =$

$$z^{r_1}\prod_{k=1}^{d_1-r_1}(z-a_{k_1})...z^{r_j}\prod_{k_j=1}^{d_j-r_j}(z-a_{k_j}).\prod_{k_{j+1}=1}^{d_{j+1}}(z-a_{k_j+1})...\prod_{k_n=1}^{d_n}(z-a_{k_n})$$ ,

we find respectively,

$$\left|F(\mathrm{Re}^{i\theta})\right| = R^{r_1+r_2+...+r_j}\prod_{k_1=1}^{d_1-r_1}\left|\mathrm{Re}^{i\theta}-a_{k_1}\right|...\prod_{k_j=1}^{d_j-r_j}\left|\mathrm{Re}^{i\theta}-a_{k_j}\right|.\prod_{k_{j+1}=1}^{d_{j+1}}\left|\mathrm{Re}^{i\theta}-a_{k_{j+1}}\right|...\prod_{k_n=1}^{d_n}\left|\mathrm{Re}^{i\theta}-a_{k_n}\right|$$

and

$$M'_F = \underset{|z|=R}{Max}\left|F(z)\right| \le$$

$$R^{r_1+r_2+...+r_j}\prod_{k_1=1}^{d_1-r_1}\left|\mathrm{Re}^{i\theta}+a_{k_1}\right|...\prod_{k_j=1}^{d_j-r_j}\left|\mathrm{Re}^{i\theta}+a_{k_j}\right|.\prod_{k_{j+1}=1}^{d_{j+1}}\left|\mathrm{Re}^{i\theta}+a_{k_{j+1}}\right|...\prod_{k_n=1}^{d_n}\left|\mathrm{Re}^{i\theta}+a_{k_n}\right|,$$

and then Lemma 1 and Proof of Theorem 2 are used .

**Theorem 3.** Let $4m_1, 4m_2, ..., 4m_n$ be the degrees of polynomial functions

$f_1, f_2, ..., f_n \in UT^4_{2,2}P(a)$, respectively and let

$f_1(\mp a) = \mp a^{q_1}$, $f_2(\mp a) = \mp a^{q_2}$, ..., $f_n(\mp a) = \mp a^{q_n}$ . Then

**(i)** if $Min(q_1, q_2, ..., q_n) \ge 3$, then

$$M'_{f_1 \cdot f_2 \cdots f_n} \ge k_3 \,.\, M'_{f_1}.M'_{f_2}...M'_{f_n} \qquad (9)$$

where

$$k_3 = \left(Sin\frac{2}{n}\frac{\pi}{8(4m_1-2)}\right)^{4m_1-2}.\left(Sin\frac{2}{n}\frac{\pi}{8(4m_2-2)}\right)^{4m_2-2}...\left(Sin\frac{2}{n}\frac{\pi}{8(4m_n-2)}\right)^{4m_n-2}.$$

**(ii)** if $q_i = 1$ ($i = 1,2,...,n$) ($0, -a, a$ are fixed points)

or

$Min(q_1, q_2, ..., q_n) = 1$,

then

$$M'_{f_1 \cdot f_2 \cdots f_n} \ge k_4 \,.\, M'_{f_1}.M'_{f_2}...M'_{f_n} \qquad (10)$$

where

$$k_4 = \left(Sin\frac{2}{n}\frac{\pi}{8(4m_1-1)}\right)^{4m_1-1}.\left(Sin\frac{2}{n}\frac{\pi}{8(4m_2-1)}\right)^{4m_2-1}...\left(Sin\frac{2}{n}\frac{\pi}{8(4m_n-1)}\right)^{4m_n-1}.$$

**Proof.** (i) By Hypothesis, knowing the form of $f_i(z)$, we can write each $f_i(z)$ under the form

$f_i(z) = z^2.$ [redacted] $^{q_i-2} + \rho_1(z^2-a^2) + \rho_2(z^6-a^4.z^2) + ... + (z^{4m_i-2} - a^{2m_i}.z^{2m_i-2})$ .

For each $f_i(z)$, $z = 0$ is a second degree zero. To conclude, just take $d_i = 4m_i$ and $r_i = 2$ ($i = 1,2,...,n$) in Formula (7) in Theorem 2. (ii) By Hypothesis, each $f_i(z)$ can be written in the form

$f_i(z)=z.$[redacted]$^{q_i-1}+\rho_1(z^3-a^2.z)+\rho_2(z^7-a^4.z^3)+...+(z^{4m_i-1}-a^{2m_i}.z^{2m_i-1})$. For each $f_i(z)$, $z=0$ is a simple zero. Again to conclude, just set $d_i=4m_i$, $r_i=1$ ($i=1,2,...,n$) in Formula (7) in Theorem 2.

**Corollary 2.** If for the first $j\le n$ fonctions $f_1,f_2,...,f_j$, $Min(q_1,q_2,...,q_n)\ge 3$ and for the remaining $n-j$ functions $f_{j+1},f_{j+2},...,f_n$, $q_i=1$
($i=1,2,...,n$) or $Min(q_1,q_2,...,q_n)=1$,
then

$$M'_{f_1\cdot f_2\cdots f_j\cdot f_{j+1}\cdots f_n}\ge k_5\ .\ M'_{f_1}.M'_{f_2}...M'_{f_j}.M'_{f_{j+1}}...M'_{f_n} \tag{11}$$

where

$$k_5=(Sin\frac{2}{n}\frac{\pi}{8(4m_1-2)})^{4m_1-2}\ .\ (Sin\frac{2}{n}\frac{\pi}{8(4m_2-2)})^{4m_2-2}...(Sin\frac{2}{n}\frac{\pi}{8(4m_j-2)})^{4m_j-2}$$

$$.\ (Sin\frac{2}{n}\frac{\pi}{8(4m_{j+1}-1)})^{4m_{j+1}-1}\ .\ (Sin\frac{2}{n}\frac{\pi}{8(4m_{j+2}-1)})^{4m_{j+2}-1}...(Sin\frac{2}{n}\frac{\pi}{8(4m_n-1)})^{4m_n-1}$$

## 5. NEW MAXIMUM MODULUS INEQUALITIES IN $UT^4_{2,2}P(a)$ ($UT^4_{2,2}P(a)$ İÇİNDE YENİ MAKSİMUM MODUL EŞİTSİZLİKLERİ )

In this section we produce new maximum modulus inequalities independently from Section 2. Let $n\ge 2$, $m_1,m_2,...,m_n$ be positive integers; let $m_1+m_2+...+m_n=p_n$ and $m_1+m_2+...+m_j=p_j$ for $j\le n$. Let 0<$R$<∞ and define the number $\lambda$ by $\lambda=\frac{1}{R^{4p_n}}$.

**Theorem 4.** Let $4m_1,4m_2,...,4m_n$ be the degrees of polynomial functions
$f_1,f_2,...,f_n\in UT^4_{2,2}P(a)$ respectively, with $f_i(a)=a^{q_i}$, $f_i(-a)=-a^{q_i}$ and $\rho_1\ne 0$ ($i=1,2,...,n$).

Then

**(i)** if $Min(q_1,q_2,...,q_n)\ge 3$, then

$$M'_{f_1\cdot f_2\cdots f_n}\ge \beta_1\ .\ M'_{f_1}.M'_{f_2}...M'_{f_n} \tag{12}$$

where

$$\beta_1=\lambda.\frac{|a|^{q_1+q_2+...+q_n}}{2^{4p_n-2n}}.$$

**(ii)** if $q_i=1$ ($i=1,2,...,n$) ($0,-a,a$ are fixed points), then

$$M'_{f_1\cdot f_2\cdots f_n}\ge \beta_2\ .\ M'_{f_1}.M'_{f_2}...M'_{f_n} \tag{13}$$

where

$$\beta_2 = \lambda.\frac{|a|^n}{2^{4p_n - n}}.$$

**(iii)** if $Min(q_1, q_2, ..., q_n) = 1$, then

$$M'_{f_1 \cdot f_2 \cdots f_n} \geq \beta_2 \, . \, M'_{f_1}.M'_{f_2}...M'_{f_n} \qquad (14)$$

where

$$\beta_3 = \lambda.\frac{|a|^{q_1+q_2+...+q_n}}{2^{4p_n - n}}.$$

**Proof.** (i) Each $f_i \in UT^4_{2,2}P(a)$ $(i = 1,2,...,n)$ can be written as

$f_i(z) = z^2.$ [redacted] $^{q_i-2} + \rho_1(z^2 - a^2) + \rho_2(z^6 - a^4.z^2) + ... + (z^{4m_i-2} - a^{2m_i}.z^{2m_i-2})$.

The polynomial in [ ] is of degree $4m_i - 2$. Thus given $|a_k| \leq R$, $(k = 1,2,...,4m_i - 2)$, for every $z, a_k \in C$ we have $|z - a_k| \leq |z| + |a_k|$, and in view of this inequality we get

$$|f_i(z)| = |z|^2.|z - a_1|.|z - a_2|...|z - a_{4m_i-2}|$$
$$\leq |z|^2.(|z| + |a_1|).(|z| + |a_2|)...(|z| + |a_{4m_i-2}|)$$
$$\leq R^2.(R + |a_1|).(R + |a_2|)...(R + |a_{4m_i-2}|)$$
$$\leq R^2.(2R).(2R)...(2R) = R^2.(2R)^{4m_i-2}$$

Now

$$M'_{f_i} = \underset{|z|=R}{Max}|f_i(z)| \leq R^2.(2R)^{4m_i-2} \ (i = 1,2,...,n),$$

and hence we obtain

$$M'_{f_1}.M'_{f_2}...M'_{f_n} \leq R^{2n}.(2R)^{4(m_1+m_2+...+m_n)} = (2)^{4p_n-2n}.\frac{1}{\lambda}. \qquad (15)$$

On the other hand, let $\Delta = +1$ or $\Delta = -1$. Then by hypothesis

$$f_1(\mp a).f_2(\mp a)...f_n(\mp a) = \Delta.a^{q_1+q_2+...+q_n}.$$

From the maximum modulus principle it follow that

$$M'_{f_1 \cdot f_2 \cdots f_n} = \underset{|z|=R}{Max}|f_1(z).f_2(z)...f_n(z)| \geq |a|^{q_1+q_2+...+q_n} \qquad (16)$$

Then (15) and (16) allow to write (17)

$$\frac{M'_{f_1 \cdot f_2 \cdots f_n}}{M'_{f_1}.M'_{f_2}...M'_{f_n}} \geq \frac{|a|^{q_1+q_2+...+q_n}}{2^{4p_n-2n}.\frac{1}{\lambda}} \qquad (17)$$

If we rearrange (17) we find (12). (ii) By Hypothesis, Each $f_i \in UT^4_{2,2}P(a)$ $(i = 1,2,...,n)$ can be writen in the form

$f_i(z) = z.$ [redacted] $^{q_i-1} + \rho_1(z^3 - a^2.z) + \rho_2(z^7 - a^4.z^3) + ... + (z^{4m_i-1} - a^{2m_i}.z^{2m_i-1})$.

Then where $|a_k| \leq R$, $(k = 1,2,...,4m_i - 1)$ for every $z \in \overline{A}$ we get

$$|f_i(z)| \leq |z|.(|z| + |a_1|).(|z| + |a_2|)...(|z| + |a_{4m_i-2}|)$$
$$\leq R.(2R).(2R)...(2R) = R.(2R)^{4m_i-1}$$

Now

$$M'_{f_i}=\underset{|z|=R}{Max}|f_i(z)|\le R^2.(2R)^{4m_i-1}\quad (i=1,2,...,n)\ ,$$

and so

$$M'_{f_1}.M'_{f_2}...M'_{f_n}\le R^{2n}.(2R)^{4(m_1+m_2+...+m_n)-n}=(2)^{4p_n-n}.\frac{1}{\lambda}\ . \qquad (18)$$

Also, in view of the hypothesis, since

$$f_1(\mp a).f_2(\mp a)...f_n(\mp a)=\Delta.a^n\ ,$$

we find that

$$M'_{f_1\cdot f_2\cdots f_n}=\underset{|z|=R}{Max}|f_1(z).f_2(z)...f_n(z)|\ge|a|^n \qquad (19)$$

Now(18) and (19) give the desired result. (iii) By Hypothesis, each $f_i\in UT^4_{2,2}P(a)$ $(i=1,2,...,n)$ can be writen as

$$f_i(z)=z.[z^{q_i-1}+\rho_1(z^3-a^2.z)+\rho_2(z^7-a^4.z^3)+...+(z^{4m_i-1}-a^{2m_i}.z^{2m_i-1})].$$

The expression in [ ] is a polynomial of degree $4m_i-1$. Then successively we obtain

$$M'_{f_1}.M'_{f_2}...M'_{f_n}\le R^n.(2R)^{4(m_1+m_2+...+m_n)-n}$$

and

$$M'_{f_1\cdot f_2\cdots f_n}=\underset{|z|=R}{Max}|f_1(z).f_2(z)...f_n(z)|\ge|a|^{q_1+q_2+...+q_n}\ . \qquad \square$$

**Corollary 3.** Let $4m_1,4m_2,...,4m_n$ be the degrees of polynomial functions $f_1,f_2,...,f_n\in UT^4_{2,2}P(a)$ respectively, with $f_i(a)=a^{q_i}$, $f_i(-a)=-a^{q_i}$ and $\rho_1\ne 0$ $(i=1,2,...,n)$. If for the $j\le n$ functions $f_1,f_2,...,f_j$, $Min(q_1,q_2,...,q_j)\ge 3$ and for the remaining $n-j$ functions $f_{j+1},f_{j+2},...,f_n$, $Min(q_{j+1},q_{j+2},...,q_n)=1$, then

$$M'_{f_1\cdot f_2\cdots f_n}\ge\beta_4\ .\ M'_{f_1}.M'_{f_2}...M'_{f_n} \qquad (20)$$

where

$$\beta_4=\lambda.\frac{|a|^{q_1+q_2+...+q_j+q_{j+1}+q_{j+2}+...+q_n}}{2^{4p_n-2j-(n-j)}}\ .$$

Now let's take

$$\alpha_1=(Sin\frac{\pi}{8(4p_j-2j)})^{4p_j-2j}.(Sin\frac{\pi}{8(4(p_n-p_j)-2(n-j))})^{4(p_n-p_j)-2(n-j)}\ ,$$

$$\alpha_2=(Sin\frac{\pi}{8(4p_j-j)})^{4p_j-j}.(Sin\frac{\pi}{8(4(p_n-p_j)-(n-j))})^{4(p_n-p_j)-(n-j)}\ ,$$

$$\alpha_3=(Sin\frac{\pi}{8(4p_j-2j)})^{4p_j-2j}.(Sin\frac{\pi}{8(4(p_n-p_j)-(n-j))})^{4(p_n-p_j)-(n-j)}\ .$$

**Theorem 5.** Given $a \neq \mp b$, Let $4m_1, 4m_2, ..., 4m_j$ be the degrees of $j$ polynomial functions $f_1, f_2, ..., f_j \in UT_{2,2}^4 P(a)$ with $f_i(a) = a^{q_i}$ , $f_i(-a) = -a^{q_i}$ and $\rho_1 \neq 0$ $(i = 1,2,..., j)$, and let $4m_{j+1}, 4m_{j+2}, ..., 4m_n$ be the degrees of $n - j$ polynomial functions $f_{j+1}, f_{j+2}, ..., f_n \in UT_{2,2}^4 P(b)$ with $f_i(b) = b^{q_i}$ , $f_i(-b) = -b^{q_i}$ and $\rho_1 \neq 0$ $(i = j+1, j+2, ..., n)$. Then

**(i)** if $Min(q_1, q_2, ..., q_n) \geq 3$, then

$$M'_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_5 . \alpha_1 \ . \ M'_{f_1} . M'_{f_2} ... M'_{f_j} . M'_{f_{j+1}} ... M'_{f_n} \tag{21}$$

where

$$\beta_5 = \lambda . \frac{|a|^{q_1+q_2+...q_j} . |b|^{q_{j+1}+q_{j+2}+...+q_n}}{2^{4p_n - 2n}} .$$

**(ii)** if $q_i = 1$ $(i = 1,2,..., n)$ $(0, -a, a$ are fixed points), then

$$M'_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_6 . \alpha_2 \ . \ M'_{f_1} . M'_{f_2} ... M'_{f_j} . M'_{f_{j+1}} ... M'_{f_n} \tag{22}$$

where

$$\beta_6 = \lambda . \frac{|a|^j . |b|^{n-j}}{2^{4p_n - n}} .$$

**(iii)** if $Min(q_1, q_2, ..., q_n) = 1$, then

$$M'_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_7 . \alpha_2 \ . \ M'_{f_1} . M'_{f_2} ... M'_{f_j} . M'_{f_{j+1}} ... M'_{f_n} \tag{23}$$

where

$$\beta_7 = \lambda . \frac{|a|^{q_1+q_2+...q_j} . |b|^{q_{j+1}+q_{j+2}+...+q_n}}{2^{4p_n - n}} .$$

**Proof.** (i) By Hypothesis, we have

$$f_1(\mp a) . f_2(\mp a) ... f_j(\mp a) = \Delta a^{q_1+q_2+...+q_j} \text{ and}$$
$$f_{j+1}(\mp b) . f_{j+2}(\mp b) ... f_n(\mp b) = \Delta b^{q_{j+1}+q_{j+2}+...+q_n} .$$

Now set $F_1 = f_1 . f_2 ... f_j$ and $F_2 = f_{j+1} . f_{j+2} ... f_n$. Then we have

$$M'_{F_1} = M'_{f_1 \cdot f_2 \cdots f_j} = \underset{|z|=R}{Max} \left| f_1(z) . f_2(z) ... f_j(z) \right| \geq |a|^{q_1+q_2+...+q_j}$$

and

$$M'_{F_2} = M'_{f_{j+1} \cdot f_{j+2} \cdots f_n} = \underset{|z|=R}{Max} \left| f_{j+1}(z) . f_{j+2}(z) ... f_n(z) \right| \geq |b|^{q_{j+1}+q_{j+2}+...+q_n}$$

and we see that

$$M'_{F_1} \ . \ M'_{F_2} = M'_{f_1 \cdot f_2 \cdots f_j} \ . \ M'_{f_{j+1} \cdot f_{j+2} \cdots f_n} \geq |a|^{q_1+q_2+...+q_j} \ . \ |b|^{q_{j+1}+q_{j+2}+...+q_n} \tag{24}$$

On the other hand, we write

$$M'_{f_1} . M'_{f_2} ... M'_{f_j} \leq R^{2j} . (R)^{4(m_1+m_2+...+m_j)-2j}$$

and

$$M'_{f_{j+1}} . M'_{f_{j+2}} ... M'_{f_n} \leq R^{2n-2j} . (R)^{4(m_{j+1}+m_{j+2}+...+m_n)-2(n-j)} .$$

From those two inequalities, we obtain

$$M'_{f_1} . M'_{f_2} ... M'_{f_n} \leq R^{2n} . (R)^{4(m_1+m_2+...+m_n)-2n} \tag{25}$$

$= 2^{4p_n-2n}.\dfrac{1}{\lambda}$ .

Now in view of (24) and (25) we find

$$M'_{F_1} \,.\, M'_{F_2} \geq \lambda.\frac{|a|^{q_1+q_2+\ldots+q_n}.|b|^{q_{j+1}+q_{j+2}+\ldots+q_n}}{2^{4p_n-2n}} \,.\, M'_{f_1 \cdot f_2 \cdots f_n} \qquad (26)$$

If we apply Formula (7) to the polynomial functions $F_1$ and $F_2$ with $n=2$, $d_1 - r_1 = 4p_j - 2j$ and $d_2 - r_2 = 4(p_n - p_j) - 2(n-j)$ we set

$$M'_{F_1.F_2} = M'_{f_1 \cdot f_2 \cdots f_n} \geq \alpha_1 \,.\, M'_{F_1} \,.\, M'_{F_2} \qquad (27)$$

Consequently,(21) follows from (26) and (27).

Parts (ii) and (iii) can be shown by similar arguments.

**Corollary 4.** Given $a \neq \mp b$, let $4m_1, 4m_2, \ldots, 4m_j$ be the degrees of $j$ polynomial functions $f_1, f_2, \ldots, f_j \in UT^4_{2,2}P(a)$ with $f_i(a) = a^{q_i}$ , $f_i(-a) = -a^{q_i}$ and $\rho_1 \neq 0$ ($i = 1,2,\ldots, j)$, and let $4m_{j+1}, 4m_{j+2}, \ldots, 4m_n$ be the degrees of $n-j$ polynomial functions $f_{j+1}, f_{j+2}, \ldots, f_n \in UT^4_{2,2}P(b)$ with $f_i(b) = b^{q_i}$ , $f_i(-b) = -b^{q_i}$ and $\rho_1 \neq 0$ ($i = j+1, j+2, \ldots, n)$ . Then

**(a)** if $Min(q_1, q_2, \ldots, q_j) \geq 3$ and $q_k = 1$ ($k = j+1, j+2, \ldots, n)$ ($0, -a, a$ are fixed points), then

$$M'_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_8 .\, \alpha_3\, M'_{f_1}.M'_{f_2} \ldots M'_{f_j}.M'_{f_{j+1}}.M'_{f_{j+2}} \ldots M'_{f_n} \qquad (28)$$

where

$$\beta_8 = \lambda.\frac{|a|^{q_1+q_2+\ldots q_j}.|b|^{n-j}}{2^{4p_n-2j-(n-j)}} .$$

**(b)** if $Min(q_1, q_2, \ldots, q_j) \geq 2$ and $Min(q_{j+1}, q_{j+2}, \ldots, q_n) = 1$, then

$$M'_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_9 .\alpha_3\, M'_{f_1}.M'_{f_2} \ldots M'_{f_j}.M'_{f_{j+1}}.M'_{f_{j+2}} \ldots M'_{f_n} \qquad (29)$$

where

$$\beta_9 = \lambda.\frac{|a|^{q_1+q_2+\ldots q_j}.|b|^{q_{j+1}+q_{j+2}+\ldots+q_n}}{2^{4p_n-2j-(n-j)}} .$$

**(c)** if $q_i = 1$ ($i = 1,2,\ldots, j)$ ($0, -a, a$ are fixed points), and $Min(q_{j+1}, q_{j+2}, \ldots, q_n) = 1$, then

$$M'_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_{10} .\alpha_2\, M'_{f_1}.M'_{f_2} \ldots M'_{f_j}.M'_{f_{j+1}}.M'_{f_{j+2}} \ldots M'_{f_n} \qquad (30)$$

where

$$\beta_{10} = \lambda.\frac{|a|^{j}.|b|^{q_{j+1}+q_{j+2}+\ldots+q_n}}{2^{4p_n-n}} .$$

**4.1. Special Case: $A$ Unit Disc Case**
**(Özel Durum: $A$ Birim Disk Durumu)**

**Lemma 3.** Let $\lambda = \frac{1}{R^{4p_n}}$ where $n \geq 2$, $m_1, m_2, ..., m_n$ are positive integres, $m_1 + m_2 + ... + m_n = p_n$ and $0 < R < \infty$. Then $\lambda = 1$ if and only if $R = 1$.

**Proof.** It is obvious.

By means of this lemma, we can state immediately the following theorems and corollaries.

**Theorem 6.** Let the hypothesis of Theorem 4 be satisfied.Then

**(i)** if $Min(q_1, q_2, ..., q_n) \geq 3$, then

$$M_{f_1 \cdot f_2 \cdots f_n} \geq \beta_1' . M_{f_1} . M_{f_2} ... M_{f_n} \tag{31}$$

where

$$\beta_1' = \frac{|a|^{q_1+q_2+...+q_n}}{2^{4p_n-2n}}.$$

**(ii)** if $q_i = 1$ ($i = 1,2,...,n$) ($0, -a, a$ are fixed points), then

$$M_{f_1 \cdot f_2 \cdots f_n} \geq \beta_2' . M_{f_1} . M_{f_2} ... M_{f_n} \tag{32}$$

where

$$\beta_2' = \frac{|a|^n}{2^{4p_n-n}}.$$

**(iii)** if $Min(q_1, q_2, ..., q_n) = 1$, then

$$M_{f_1 \cdot f_2 \cdots f_n} \geq \beta_3' . M_{f_1} . M_{f_2} ... M_{f_n} \tag{33}$$

where

$$\beta_3' = \frac{|a|^{q_1+q_2+...+q_n}}{2^{4p_n-n}}.$$

**Corollary 5.** Let the conditions of Corollary 3 be satisfied. Then

$$M_{f_1 \cdot f_2 \cdots f_n} \geq \beta_4' . M_{f_1} . M_{f_2} ... M_{f_n} \tag{34}$$

where

$$\beta_4' = \frac{|a|^{q_1+q_2+...+q_j+p+q_{p+1}+q_{p+2}+...+q_n}}{2^{4p_n-2j-(n-j)}}.$$

**Theorem 7.** Let the clauses of Theoerm 5 be satisfied. Then

**(i)** if $Min(q_1, q_2, ..., q_n) \geq 3$, then

$$M_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_5' \cdot \alpha_1 M_{f_1} . M_{f_2} ... M_{f_j} . M_{f_{j+1}} . M_{f_{j+2}} ... M_{f_n} \quad (35)$$

where

$$\beta_5' = \frac{|a|^{q_1+q_2+...q_j} . |b|^{q_{j+1}+q_{j+2}+...+q_n}}{2^{4p_n-2n}}.$$

**(ii)** if $q_i = 1$ ($i = 1,2,...,n$) ($0, -a, a$ are fixed points), then

$$M_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_6' . \alpha_2 M_{f_1} . M_{f_2} ... M_{f_j} . M_{f_{j+1}} . M_{f_{j+2}} ... M_{f_n} \quad (36)$$

where

$$\beta_6' = \frac{|a|^j . |b|^{n-j}}{2^{4p_n-n}}.$$

**(iii)** if $Min(q_1, q_2, ..., q_n) = 1$, then

$$M_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_7' . \alpha_2 M_{f_1} . M_{f_2} ... M_{f_j} . M_{f_{j+1}} . M_{f_{j+2}} ... M_{f_n} \quad (37)$$

where

$$\beta_7' = \frac{|a|^{q_1+q_2+...q_j} . |b|^{q_{j+1}+q_{j+2}+...+q_n}}{2^{4p_n-n}}.$$

**Corollary 6.** Let the clauses of Corolary 4 be met. Then

**(a)** if $Min(q_1, q_2, ..., q_j) \geq 3$ and $q_k = 1$ ($k = j+1, j+2, ..., n$), then

$$M_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_8' . \alpha_3 M_{f_1} . M_{f_2} ... M_{f_j} . M_{f_{j+1}} . M_{f_{j+2}} ... M_{f_n} \quad (38)$$

where

$$\beta_8' = \frac{|a|^{q_1+q_2+...q_j} . |b|^{n-j}}{2^{4p_n-2j-(n-j)}}.$$

**(b)** if $Min(q_1, q_2, ..., q_j) \geq 3$ and $Min(q_{j+1}, q_{j+2}, ..., q_n) = 1$, then

$$M_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_9' . \alpha_3 M_{f_1} . M_{f_2} ... M_{f_j} . M_{f_{j+1}} . M_{f_{j+2}} ... M_{f_n} \quad (39)$$

where

$$\beta_9' = \frac{|a|^{q_1+q_2+...q_j} . |b|^{q_{j+1}+q_{j+2}+...+q_n}}{2^{4p_n-2j-(n-j)}}.$$

**(c)** if $q_i = 1$ ($i = 1,2,...,j$) ($0, -a, a$ are fixed points), and $Min(q_{j+1}, q_{j+2}, ..., q_n) = 1$, then

$$M_{f_1 \cdot f_2 \cdots f_j \cdot f_{j+1} \cdots f_n} \geq \beta_{10}' . \alpha_2 M_{f_1} . M_{f_2} ... M_{f_j} . M_{f_{j+1}} . M_{f_{j+2}} ... M_{f_n} \quad (40)$$

where

$$\beta'_{10} = \frac{|a|^{j}.|b|^{q_{j+1}+q_{j+2}+...+q_n}}{2^{4p_n - n}}$$ .

## 6. RESULTS AND RECOMMENDATIONS (SONUÇ VE ÖNERİLER)

**6.1.** For the elements of $UT_{2,2}^{4}P(a)$ with a) $\rho_1 = 0$, b) $\rho_1 = \rho_2 = 0$ ,…, c) $\rho_1 = \rho_2 = ... = \rho_l = 0$ ($l \le m$) we can formulate inequalities.

**6.2.** The maximum modulus of $p(z)$ on circle $|z| = R$ ($R > 1$) we have from [9 and 10]

$$\max_{|z|=R} |p(z)| \le R^n \max_{|z|=1} |p(z)| \quad (41)$$

If $p(z) \ne 0$ in $|z| < 1$, then from [1 and 10]

$$\max_{|z|=R} |p(z)| \le (\frac{R^n + 1}{2}) \max_{|z|=1} |p(z)| , \ (R > 1) \quad (42)$$

If we take $p(z) = f_1(z).f_2(z)...f_n(z)$ $(n \ge 2)$ in Formulas (41) and (42), we have
$\deg(p(z) = 4p_n$ and we can write a generalization of (41) and (42) respectively

$$M'_{f_1 \cdot f_2 \cdots f_n} \le R^{4p_n} . M_{f_1 \cdot f_2 \cdots f_n} , \ (R > 1) \quad (43)$$

and

$$M'_{f_1 \cdot f_2 \cdots f_n} \le (\frac{R^{4p_n} + 1}{2}) . M_{f_1 \cdot f_2 \cdots f_n} , \ (R > 1) . \quad (44)$$

Now consider Theorem 4 and Theorem 6 Let $|a| = 1$ and ($R > 1$). Then

**a)** if $Min(q_1, q_2, ..., q_n) \ge 3$, then

$$\frac{M'_{f_1}.M'_{f_2}...M'_{f_n}}{M'_{f_1 \cdot f_2 \cdots f_n}} \le \frac{1}{\lambda \beta_1'^2} . \frac{M_{f_1 \cdot f_2 \cdots f_n}}{M_{f_1}.M_{f_2}...M_{f_n}} \quad (45)$$

**b)** if $q_i = 1$ ($i = 1,2,...,n)$ ($0, -a, a$ are fixed points), then

$$\frac{M'_{f_1}.M'_{f_2}...M'_{f_n}}{M'_{f_1 \cdot f_2 \cdots f_n}} \le \frac{1}{\lambda \beta_2'^2} . \frac{M_{f_1 \cdot f_2 \cdots f_n}}{M_{f_1}.M_{f_2}...M_{f_n}} \quad (46)$$

**c)** if $Min(q_1, q_2, ..., q_n) = 1$, then

$$\frac{M'_{f_1}.M'_{f_2}...M'_{f_n}}{M'_{f_1 \cdot f_2 \cdots f_n}} \le \frac{1}{\lambda \beta_3'^2} . \frac{M_{f_1 \cdot f_2 \cdots f_n}}{M_{f_1}.M_{f_2}...M_{f_n}} \quad (47)$$

If $|a|=1$ in Formulas (45),(46)and(47), then since $\beta_1'=\frac{1}{2^{4p_n-2n}}$ and $\beta_2'=\beta_3'=\frac{1}{2^{4p_n-n}}$,

**a)** if $Min(q_1,q_2,...,q_n)\geq 3$, then

$$\frac{M'_{f_1}.M'_{f_2}...M'_{f_n}}{M'_{f_1\cdot f_2\cdots f_n}}\leq\frac{(2R)^{4p_n}}{2^{2n}}\cdot\frac{M_{f_1\cdot f_2\cdots f_n}}{M_{f_1}.M_{f_2}...M_{f_n}} \quad (48)$$

**b)** if $q_i=1$ ($i=1,2,...,n$) or $Min(q_1,q_2,...,q_n)=1$, then

$$\frac{M'_{f_1}.M'_{f_2}...M'_{f_n}}{M'_{f_1\cdot f_2\cdots f_n}}\leq\frac{(2R)^{4p_n}}{2^{n}}\cdot\frac{M_{f_1\cdot f_2\cdots f_n}}{M_{f_1}.M_{f_2}...M_{f_n}}. \quad (49)$$

**6.3.** Besides, we can denote by $NUTP(a)$ non univalent polynomial functions $f:C\rightarrow C$ with

$f(0)=0$, $f(a)=a^q$, $f(-a)=a^q$.

In $NUTP(a)$ we can write the sets

$NUT_{2,2}^4P(a)$, $NUT_{3,1}^4P(a)$, $NUT_{1,3}^4P(a)$ , $NUT_{4,0}^4P(a)$

for the polynomial of degree $4m$ and $q=2,4,...,4m-2$, and formulate Maximum Modulus inequalities.

## 7. DISCUSSIONS (TARTIŞMALAR)

The coefficients in the inequalities found in Ostrowski [12] , Rassias [13], Mohr [11] , Çelik [6] and Çelik [5] don't depend on fixed points of polynomial functions or on the powers of those points. But when $A$ is the unit disc in the found inequalities for $UT_{2,2}^4P(a)$ new coefficients depend on the power of the point $|a|$. Moreover, when $A$ is a disc with arbitrary radius $R<\infty$ , it also depend on $\lambda$ ( naturally on $R$ ). Thus as the parameters remain invariant except $|a|$ and $R$, we can choose $|a|$ and $R$ in such a way that the coefficients $\beta_i$ ($i=1,2,3,4$) produce results better than the coefficients $k$ (or $\dot{k}$) and $k_i$ ($i=1,2,3,4,5$). Given $R$, the greatest coefficient $\beta_i$ ($i=1,2,3,4$) is obtained when $|a|=R$, and the smallest $\beta_i$ ($i=1,2,3,4$) is found when $|a|$ is as close as to the origin. As to $\beta_j$ ($j=5,6,...,10$), the greatest is obtained for $|a|=R$ and $|b|=R$, on the smallest one when $|a|$ and $|b|$ are closest to the origin. Consequently, when working in $UT_{2,2}^4P(a)$ which coefficients are suitable is a matter of taste. Similar interpretation can be given in case $A$ is the unit disc.

**REFERENCES (KAYNAKLAR)**

1. Ankeny, N.C. and Rivlin, T.J., (1955). On a theorem of S. Brenstein, Pasific J. Math., 849-852.
2. Avcı, Y. and Zlotkiewicz, E., (1997). On univalent functions with three preassigend values, Tr. J. of Mathematics, 21, 15-23.
3. Çelik, A., (2012). New inequalities for Maximum Modulus Values of polynomial functions, Hacettepe Journal of Mathematics and Statistics, volume 41 (2), 255-263.
4. Çelik, A., (2009). On the "univalent functions with three preassigend Values and automorphisms of an open disc",E-Journal New World Sciences Academy,volume 4,number 2, 36-41.
5. Çelik, A., (2004). Maximum module values of polynomials on $|z| = R$ ($R > 1)$, Üniv. Beograd, publ. Elektrotehn. Fak.,ser. Mat.15,1-6.
6. Çelik, A., (1997). A note on Mohr's paper, Üniv. Beograd, publ. Elektrotehn. Fak.,ser. Mat.8, 51-54.
7. Deshpande, J.V., (1986). Complex Analysis, Tata MCGraw-Hill Publising Company, New Delhi.
8. Duren, P.L., (1983). Univalent functions Newyork-Berlin, Springer Verlag.
9. Milonovic' G.V., Mitrinovic' D.S., and Rassias, M.TH., (1994). Extremal Problems, Inequalities Zeros ,Word Scientific Publ. Co., Singapore, New Jersey, London.
10. Mir, A., Devan, K.K., and Sing, N., (2009). Some inequalities concerning The rate of growty of polinomials, Turk. J. Math., 33, 239-247.
11. Mohr, E., (1992). Bemerkung Zu der arbeit Van A.M. Ostrowski Notiz uber Maximalwerte von polynomen auf dem einheitskreis Üniv. Beograd, publ. Elektrotehn.Fak,ser. Mat.3, 3-4.
12. Ostrowski, A.M., (1979). Notiz uber Maximalwerte von polynomen auf dem einheitskreis, Üniv. Beograd, publ.Elektrotehn. Fak., ser. Mat. Fiz.,No 634-637, 55-56.
13. Rassias, M.TH., (1986). A new inequality for complex-valued polynomial functions, Proc. Amer. Math. Soc. 9, 296-298.