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ON $(N,p,q)C_1$ SUMMABILITY OF THE SEQUENCE $\{nB_n(x)\}$

ABSTRACT

In this article a generalization of a theorem on the $(N,p)C_1$ summability of derived series proved by Sharma is established for generalized $(N,p,q)C_1$ summability of the sequence $\{nB_n(x)\}$.

Keywords: Nörlund Summability, Generalized (N,p,q) Summability, Cesaro Summability, Fourier Series, Derived Series

$\{nB_n(x)\}$ DİZİNİN $(N,p,q)C_1$ TOPLANABİLİRLİĞİ ÜZERİNE

ÖZET

Bu makalede Sharma tarafından türev serilerinin $(N,p)C_1$ toplanabilirliği üzerine ispatlanan teoremin genelleştirilmiş $(N,p,q)C_1$ toplanabilirliği üzerine bir genelleştirilmesi yapılmıştır.

Anahtar kelimeler: Nörlund toplanabilirliği, Genelleştirilmiş Nörlund Toplanabilirliği, Cesaro toplanabilirliği, Fourier Serileri, Türev serileri

1. INTRODUCTION (GİRİŞ)

Various types of criteria, under varying conditions, for the Nörlund summability of the derived Fourier series have been obtained by Hille and Tamarkin [4], Astrachan [1] and Prasad and Siddiqi [9]. Then, Sharma [11] and Prasad [10] studied about Nörlund summability of derived series. In 2001, Lal and Yadav [5] studied about the summability of derived series of Fourier series on $(N,p,q)C_1$ summability. In this article, we generalized Sharma's [11] theorem about derived series of summability on $(N,p)C_1$ as generalized $(N,p,q)C_1$. The definition and notations which will be used in proof of the theorem is followed.

Definitions and Notations:

Definition 1. Let $\sum u_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of real constants, and let us write

$$P_n = \sum_{k=0}^{n} p_k \; ; P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v} s_v \quad ; (P_n \neq 0)$$

defines the sequence of Nörlund means of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. The series $\sum u_n$ is said to be summable (N,p) to the sum s, if $\lim_{n\to\infty} t_n = s$ [13].

Definition 2. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of nth partial sums $\{s_n\}$. Let p, q denote the sequences $\{p_n\}$ and $\{q_n\}$ with $p_{-1} = 0, q_{-1} = 0,$ respectively. Given two sequences p and q, the convolution $(p * q)_n$ is defined by

$$R_n = (p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k = \sum_{k=0}^{n} p_k q_{n-k}. \quad (1)$$

When $(p * q)_n \neq 0$ for all n, for any sequence $\{s_n\}$ we write

$$t_n^{p,q} = \frac{1}{(p * q)_n}\sum_{k=0}^{n} p_{n-k} q_k s_k. \quad (2)$$

If $t_n^{p,q} \to s$ as $n \to \infty$, we write $\sum_{n=0}^{\infty} a_n = s(N,p,q)$ or $\{s_n\} \to s(N,p,q)$ [2].

Definition 3. We write $(C,1)$ means of $\sum_{n=0}^{\infty} a_n$ series or sequence $\{s_n\}$ as follows:

$$\sigma_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \quad (3)$$

If $\sigma_n \to s$ as $n \to \infty$, we write $\sum_{n=0}^{\infty} a_n = s(C,1)$ or $\{s_n\} \to s(C,1)$ [3].

Definition 4. The (N,p,q) transform of the $(C,1)$ transform C_1 defines the $(N,p,q)C_1$ transform of the partial sum $\{s_n\}$ of the series $\sum_{n=0}^{\infty} a_n$. Thus, if

$$t_n^{p,q,C_1} = \frac{1}{(p*q)_n}\sum_{k=0}^{n} p_{n-k} q_k \sigma_k \qquad (4)$$

tends to s, as $n \to \infty$ then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by $(N,p,q)C_1$ summable to s. It is denoted as, $t_n^{p,q,C_1} \to s((N,p,q)C_1)$ [5].
The necessary and sufficient conditions that the (N,p,q) method be regular are

$$\sum_{k=0}^{n} |p_{n-k} q_k| = O(|(p*q)_n|) \qquad (5)$$

and

$$p_{n-k} = o(|(p*q)_n|), as\ n \to \infty, \qquad (6)$$

for every fixed $k \geq 0$, for each $q_k \neq 0$. The $(C,1)$ summability is also regular. Let us verify the regularity conditions of $(N,p,q)C_1$ method

$$\begin{aligned} t_n^{p,q,C_1} &= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k \sigma_k \\ &= \frac{1}{R_n}\sum_{k=0}^{n} \frac{p_{n-k} q_k}{n-k+1}\sum_{v=0}^{n-k} s_v \\ &= \sum_{k=0}^{\infty} C_{n,k} s_k \end{aligned}$$

where

$$C_{n,k} = \begin{cases} \dfrac{1}{R_n}\left(\dfrac{p_{n-k} q_k}{n-k+1}\right)\displaystyle\sum_{v=0}^{k} 1, & k \leq n \\ 0, & k > n. \end{cases}$$

Now

(i) $\sum_{k=0}^{\infty} |C_{n,k}| = \frac{1}{R_n}\left(\sum_{k=0}^{n} p_{n-k} q_k\right) = 1$

(ii) $C_{n,k} = \frac{p_{n-k} q_k}{R_n} \to 0\ as\ n \to \infty, for\ fixed\ k.$

(iii) $\sum_{k=0}^{\infty} C_{n,k} = 1.$

So $(N,p,q)C_1$ method is regular [5].

Definition 5. Let $f(t)$ be a function which is integrable in the sense of Lebesque over the interval $(-\pi,\pi)$ and is defined outside this interval by periodicity. Let the Fourier series of $f(x)$ be

$$\frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n Cosnx + b_n Sinnx) = \frac{1}{2}a_0 \sum_{1}^{\infty} A_n(x) \qquad (7)$$

then the derived series of (7) is

$$\sum_{1}^{\infty} n(b_n Cosnx - a_n Sinnx) = \sum_{1}^{\infty} nB_n(x). \qquad [6].(8)$$

We shall use the following notations:

$$\psi(t) = f(x+t) - f(x) - l$$

$$\tau = [1/t], \text{ is the integral part of } \frac{1}{t}$$

$$\Psi(t) = \int_0^t |\psi(u)|\,du.$$

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this article the authors generalized Sharma's [11] theorem about Nörlund summability as generalized $(N,p,q)C_1$ summability. We hope that the result of the study will contribute for the next studies about summability.

3. MAIN RESULTS (ANA SONUÇLAR)

In 1970 Sharma [11] proved the following theorem.

- **Theorem 1:** If $\{p_n\}$ is a monotonic, non-increasing sequence of real positive constant such that

$$P_n \to \infty \ as\ n \to \infty,$$
$$logn = O(P_n), and$$
$$\int_0^t |\psi(u)|\,du = o\left(\frac{t}{P_\tau}\right) \ as\ t \to 0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N,p_n)C_1$ to the sum $\frac{l}{\pi}$ [11].

Lemma. If p_n is non-negative and non-increasing and q_n is non-negative and non-decreasing then

$$\left|\sum_{\nu=0}^{n-1} p_\nu q_{n-\nu}\cos(n-\nu)t\right| \le kR_\tau$$ [7].

Proof. We write,

$$\left|\sum_{\nu=0}^{n-1} p_\nu q_{n-\nu}\cos(n-\nu)t\right| \le \left|\sum_{\nu=0}^{n-1} p_\nu q_{n-\nu} e^{(n-\nu)t}\right|$$
$$= \left|e^{int}\sum_{\nu=0}^{n-1} p_\nu q_{n-\nu} e^{-i\nu t}\right|$$
$$= \left|\sum_{\nu=0}^{\tau-1} p_\nu q_{n-\nu} e^{-i\nu t}\right| + \left|\sum_{\nu=\tau}^{n-1} p_\nu q_{n-\nu} e^{-i\nu t}\right|$$

But

$$\left|\sum_{\nu=0}^{\tau-1} p_\nu q_{n-\nu} e^{-i\nu t}\right| \le \sum_{\nu=0}^{\tau-1} p_\nu q_{n-\nu} \le R_\tau \qquad (9)$$

And Abel's transformation

$$\left|\sum_{\nu=0}^{\tau-1} p_\nu q_{n-\nu} e^{-i\nu t}\right| \le \max_{\tau \le \nu \le n-1}\left|\sum_{k=0}^{\nu-1} e^{-ikt}\right| \times$$
$$\times \left(\sum_{\nu=\tau}^{n-2}\left(p_\nu q_{n-\nu} \quad p_{\nu+1} q_{n-(\nu+1)}\right) + p_\tau q_{n-\tau} \quad p_{n-1} q_1\right)$$
$$= 2p_\tau q_{n-\tau} \max_{\tau \le \nu \le n-1}\left|\sum_{k=0}^{\nu-1} e^{-ikt}\right|$$
$$= 2p_\tau q_{n-\tau} \max_{\tau \le \nu \le n-1}\left|\frac{1-e^{-i\nu t}}{1-e^{-it}}\right|$$
$$\le 4p_\tau q_{n-\tau}\left(\frac{1}{\sin\frac{t}{2}}\right)$$
$$\le k\frac{p_\tau q_{n-\tau}}{t}. \qquad (10)$$

From (9) and (10) we write,

$$\left|\sum_{\nu=0}^{n-1} p_\nu q_{n-\nu}\cos(n-\nu)t\right| \le R_\tau + k\frac{p_\tau q_{n-\tau}}{t} \le R_\tau + k(\tau+1)p_\tau q_{n-\tau} \le kR_\tau$$ [7].

In this paper we will obtain the following theorem.

- **Theorem 2.** If $\{p_n\}$ is a non-negative, monotonic, non-increasing sequence and $\{q_n\}$ a non-negative, non-decreasing sequence such that

$$R_n \to \infty \; as\; n \to \infty,$$
$$logn = O(R_n), and$$
$$\int_0^t |\psi(u)|du = o\left(\frac{t}{R_\tau}\right) \; as\; t \to 0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N,p,q)C_1$ summable to sum $\frac{l}{\pi}$.

Proof. Let σ_n is denoted the (C,1) transform of the sequence $\{nB_n(x)\}$

$$\sigma_n - \frac{l}{\pi} - \frac{1}{n}\sum_{k=1}^{n} kB_k(x) - \frac{l}{\pi} - \frac{1}{\pi}\int_0^\pi \psi(t)\left[\frac{sinnt}{nt^2} - \frac{cosnt}{t}\right]dt + o(1)$$

by Riemann-Lebesque theorem [8].

$$\begin{aligned}
t_n^{p,q,C_1} &= \frac{1}{R_n}\left(\sum_{\nu=1}^{n} p_{n-\nu} q_\nu \sigma_\nu\right)\\
&= \sum_{\nu=1}^{n} \frac{p_{n-\nu}q_\nu}{R_n}\frac{1}{\pi}\int_0^\pi \psi(t)\left[\frac{sin\nu t}{\nu t^2} - \frac{cos\nu t}{t}\right]dt + \frac{l}{\pi} + o(1)\\
t_n^{p,q,C_1} - \frac{l}{\pi} &= \int_0^\pi \psi(t)\left(\frac{1}{\pi}\sum_{\nu=1}^{n} \frac{p_{n-\nu}q_\nu}{R_n}\left[\frac{sin\nu t}{\nu t^2} - \frac{cos\nu t}{t}\right]dt\right) + o(1)\\
&= \int_0^\pi \psi(t)r_n(t)dt + o(1)\\
&= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi\right)\psi(t)r_n(t)dt + o(1)\\
&= I_1 + I_2 + I_3 .
\end{aligned}$$

Now

$$r_n(t) = O\left(\frac{1}{t}\right). \qquad (11)$$

Since

$$r_n(t) = O(n), \text{ when } 0 \le t \le \frac{1}{n} \qquad (12)$$

and using the condition (iii) of theorem 2, we have

$$I_1 = o(1).$$

Since the $(N,p,q)C_1$ method is regular

$$I_3 = o(1),$$

$$\begin{aligned}
I_2 &= \int_{1/n}^{\delta} \psi(t)r_n(t)dt + o(1)\\
&= \frac{1}{\pi}\int_{1/n}^{\delta}\psi(t)\sum_{\nu=1}^{n}\frac{p_{n-\nu}q_\nu}{R_n}\frac{sin\nu t}{\nu t^2}dt - \frac{1}{\pi}\int_{1/n}^{\delta}\psi(t)\sum_{\nu=1}^{n}\frac{p_{n-\nu}q_\nu}{R_n}\frac{cos\nu t}{t}dt\\
&= I_{21} - I_{22}; \text{ say.}
\end{aligned}$$

Considering Stieltjes integral, using Lemma and the conditions (i) and (ii) of the theorem 2, we have,

$$I_{22} = O\left(\frac{1}{R_n}\int_{1/n}^{\delta}\frac{|\psi(t)|}{t}R_\tau dt\right) = o(1).$$

$$I_{21} = \frac{1}{\pi}\int_{1/n}^{\delta}|\psi(t)|\sum_{v=1}^{n}\frac{p_{n-v}q_v}{R_n}\frac{\sin vt}{vt^2}dt$$

$$= \frac{1}{\pi R_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\sum_{v=0}^{n-1}p_v q_{n-v}\frac{\sin(n-v)t}{(n-v)}dt$$

$$= \frac{1}{n\pi R_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\sum_{v=0}^{n-1}p_v q_{n-v}\sin(n-v)t\left(\frac{v}{(n-v)}+1\right)dt$$

$$= \frac{1}{n\pi R_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\sum_{v=0}^{n-1}\frac{vp_v q_{n-v}\sin(n-v)t}{(n-v)}dt + \frac{1}{n\pi R_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\sum_{v=0}^{n-1}p_v q_{n-v}\sin(n-v)t$$

$= I_{211} + I_{212}$; say.

By Abel Lemma [12], we have

$$\sum_{v=0}^{n-1}\frac{vp_v q_{n-v}\sin(n-}{(n-v)} = p_{n-1}q_1\sum_{v=0}^{n-1}\frac{v\sin(n-v)t}{(n-v)} +$$

$$+\sum_{v=0}^{n-2}(p_v q_{n-v} - p_{v+1}q_{n-(v+1)})\sum_{m=0}^{v-1}\frac{m\sin(n-m)t}{n-m}$$

$$= \sum_{v=0}^{n-2}(p_v q_{n-v} - p_{v+1}q_{n-(v+1)})\left(\sum_{k=0}^{v-1}(-1)\sum_{m=0}^{k}\frac{\sin(n-m)t}{n-m} + v\sum_{k=}^{v}\right.$$

$$+p_{n-1}q_1\sum_{v=0}^{n-2}(-1)\sum_{m=0}^{v}\frac{\sin(n-m)t}{n-m} + (n-1)p_{n-1}q_1\sum_{m=0}^{n-1}\frac{\sin(n-m)t}{n-m}$$

Hence we obtain

$$I_{211} = \frac{1}{n\pi R_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\left[O\left(\sum_{v=0}^{n-2}v|p_v q_{n-v} - p_{v+1}q_{n-(v+1)}|\right) + O(n-1)p_{n-1}q_1\right]dt$$

$$= O\left(\frac{1}{n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\right)$$

$$= O\left(\frac{1}{n}\left[\frac{\Psi(t)}{t^2}\right]_{1/n}^{\delta} + \frac{2}{n}\int_{1/n}^{\delta}\frac{\Psi(t)}{t^3}dt\right)$$

$$= o(1) + o\left(\frac{1}{n}\int_{1/n}^{\delta}\frac{1}{t^2}dt\right)$$

$= o(1)$

and

$$I_{212} = \frac{1}{n\pi R_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\sum_{v=0}^{n-1}p_v q_{n-v}\sin(n-v)t\,dt$$

$$= O\left(\frac{1}{nR_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\sum_{v=0}^{n-1}p_v q_{n-v}\right)$$

$$= O\left(\frac{1}{n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\right).$$

same as I_{211},

$I_{212} = o(1)$.

This completes the proof of the theorem 2.

4. CONCLUSION AND SUGGESTIONS (SONUÇ VE ÖNERİLER)

In this article, we have successfully established a theorem for $(N,p,q)C_1$ summability of the sequence $\{nB_n(x)\}$. It may be observed that this theorem can be considered similar to the theorem of Lal & Yadav [5] for derived Fourier series. In the next studies, this theorem can be rearranged for almost Nörlund summability or conjugate derived Fourier series.

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