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$Heis^3$ LORENTZIAN HEISENBERG GRUBUNDA BİMİNİMAL EĞRİLER VE YÜZEYLER

ÖZET

Bu makalede, $Heis^3$ Lorentzian Heisenberg grubunda biminimal eğriler ve yüzeyler çalışıldı. Lorentzian Heisenberg grubunda non geodezik biminimal eğriler karakterize edildi ve biminimal yüzeyler için yeni bir örnek oluşturuldu.
Anahtar Kelimeler: Heisenberg Grup, Biminimal Eğri, Biminimal Yüzey, Lorentzian Metrik, Biminimallik

BIMINIMAL CURVES AND SURFACES OF THE LORENTZIAN HEISENBERG GROUP $Heis^3$

ABSTRACT

In this paper, we study biminimal curves and surfaces in the Lorentzian Heisenberg group $Heis^3$. We characterize non-geodesic biminimal curves and construct new example of biminimal surfaces of the Lorentzian Heisenberg group $Heis^3$.
Keywords: Heisenberg Group, Biminimal Curve, Biminimal Surface, Lorentzian Metric, Biminimality

1. GİRİŞ (INTRODUCTION)

Recently, important progress has been made in the study of both the geometry and the analytic properties of biharmonic maps. In differential geometry, a special attention has been payed to the study of biharmonic submanifolds, i.e. submanifolds such that the inclusion map is a biharmonic map.

In [1] the authors found new examples of biharmonic maps by conformally deforming the domain metric of harmonic ones. In this vein, new examples of biharmonic maps between the n-dimensional Euclidean sphere and the (n+1)-dimensional sphere endowed with a special metric, conformally equivalent to the canonical one, were constructed in [4], while in [2] the author analyzed the behavior of the biharmonic equation under the conformal change of metric on the target manifold of harmonic Riemannian submersions. Moreover, in [5] the author gave some extensions of the results in [2] together with some further constructions of biharmonic maps.

Let $f:(M,g)\to(N,h)$ be a smooth map between two Lorentzian manifolds. The bienergy $E_2(f)$ of f over compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_\Omega h(\tau(f),\tau(f))dv_g,$$

where $\tau(f)=\mathrm{trace}_g\nabla df$ is the tension field of f and dv_g is the volume form of M. Using the first variational formula one sees that f is a biharmonic map if and only if its bitension field vanishes identically, i.e.,

$$\tau_2(f):=-\Delta^f(\tau(f))-\mathrm{trace}_g R^N(df,\tau(f))df=0, \qquad (1.1)$$

where

$$\Delta^f=-\mathrm{trace}_g(\nabla^f)^2=-\mathrm{trace}_g\left(\nabla^f\nabla^f-\nabla^f_{\nabla^M}\right)$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN)$ and R^N is the curvature operator of (N,h) defined by

$$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z. \qquad (1.2)$$

An isometric immersion $f:(M,g)\to(N,h)$ is called a biminimal immersion if it is a critical point of the bienergy functional E_2 with respect to all normal variation with compact support. Here, a normal variation means a variation $\{f_t\}$ through $f=f_0$ such that the variational vector field $V=df_t/dt|_{t=0}$ is normal to M.

The Euler-Lagrange equation of this variational problem is $\tau_2(f)^\perp=0$. Here $\tau_2(f)^\perp$ is the normal component of $\tau_2(f)$.

An isometric immersion $f:M\to N$ is called a $\lambda-$*biminimal* immersion if it is a critical point of the functional:

$$E_{2,\lambda}(f)=E_2(f)+\lambda E(f),\ \lambda\in\mathbb{R}. \qquad (1.3)$$

The Euler-Lagrange equation for $\lambda-$*biminimal* immersions is

$$\tau_2(f)^\perp=\lambda\tau(f). \qquad (1.4)$$

We know that an immersion free biminimal if it is biminimal for $\lambda=0$

[3]. In the instance of an isometric immersion $f : M \to N$, the biminimal condition is

$$\Delta \mathbf{H} - traceR^N (df, \mathbf{H})df]^{\perp} + \lambda \mathbf{H} = 0, \quad (1.5)$$

where $\mathbf{H}$ = $H\mathbf{N}$ its mean curvature vector and H the mean curvature function.

On the other hand, in [5], E. Loubeau and S. Montaldo introduced the notion of biminimal immersion.

Proposition 1.1. [5] Let $f : M^n \to N^{n+1}$ be an isometric immersion of codimension-one and $\mathbf{H}$ = $H\mathbf{N}$ its mean curvature vector. Then is f is biminimal if and only if:

$$\Delta^M H = (\| B \|^2 - Ricci(\mathbf{N}))H, \quad (1.6)$$

where B the second fundamental form of , $\mathbf{N}$ a unit normal vector field to $f(M) \subset N$ and $\mathbf{H}$ = $H\mathbf{N}$ its mean curvature vector field of f (H the mean curvature function).

An isometric immersion $f : (M, g) \to (N, h)$ is said to be biminimal if it is a critical point of the bienergy functional under all normal variations. Thus the biminimality is weaker than biharmonicity for isometric immersions, in general.

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this paper, we study biminimal curves and surfaces in the Lorentzian Heisenberg group $Heis^3$ and we characterize non-geodesic biminimal curves and construct new example of biminimal surfaces of the Lorentzian Heisenberg group $Heis^3$.

3. $Heis^3$ HEİSENBERG GRUBUNDA SOL İNVARİANT LORENTZİAN METRİK (LEFT INVARIANT LORENTZIAN METRIC IN HEISENBERG GROUP $Heis^3$)

The Lorentzian Heisenberg group $Heis^3$ can be seen as the space $\mathbb{R}^3$ endowed with the multiplication

$$(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \bar{x}y + x\bar{y}). \quad (3.1)$$

$Heis^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by

$$g = -dx^2 + dy^2 + (xdy + dz)^2. \quad (3.2)$$

where

$$\omega^1 = dz + xdy, \quad \omega^2 = dy, \quad \omega^3 = dx$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}. \quad (3.3)$$

The corresponding Lie brackets are

$$[e_2, e_3] = 2e_1, \quad [e_1, e_3] = [e_1, e_2] = 0$$

with

$$g(e_1, e_1) = g(e_2, e_2) = 1, \ g(e_3, e_3) = -1.$$

- **Proposition 3.1.** For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:

$$\nabla = \begin{pmatrix} 0 & e_3 & e_2 \\ e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{pmatrix},$$

where the (i,j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{e_k, k=1,2,3\} = \{e_1, e_2, e_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Moreover we put

$$Rabc = R(e_a, e_b)e_c, \; R_{abcd} = R(e_a, e_b, e_c, e_d),$$

where the indices a,b,c,d take the values $1,2$ and 3.

$$R_{121} = e_2, \quad R_{131} = e_3, R_{122} = -e_1,$$
$$R_{232} = -3e_3, R_{133} = e_1, R_{233} = -3e_2,$$

and

$$R_{1212} = 1, R_{1313} = -1, R_{2323} = 3. \quad (3.4)$$

4. Heis^3 LORENTZIAN HEISENBERG GRUBUNDA BİMİNİMAL EĞRİLER (BIMINIMAL CURVES IN LORENTZIAN HEISENBERG GROUP Heis^3)

Let $\gamma : I \to Heis^3$ be a timelike curve on Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{T,N,B\}$ be the Frenet frame fields tangent to Lorentzian Heisenberg group $Heis^3$ along γ defined as follows: T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ), and B is chosen so that $\{T,N,B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_T T &= kN \\ \nabla_T N &= kT + \tau B \\ \nabla_T B &= -\tau N, \end{aligned} \quad (4.1)$$

where $k = |\tau(\gamma)| = |\nabla_T T|$ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{aligned} T &= T_1 e_1 + T_2 e_2 + T_3 e_3, \\ N &= N_1 e_1 + N_2 e_2 + N_3 e_3, \\ B &= T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3. \end{aligned}$$

- **Theorem 4.1.** Let $\gamma : I \to Heis^3$ be a non-geodesic timelike curve on Lorentzian Heisenberg group Heis^3 parametrized by arc length. Then γ is a timelike non-geodesic biminimal curve if and only if

$$k'' + k^3 - k\tau^2 = k(1-4B_1^2),$$
$$2\tau k' + k\tau' = 2kN_1B_1.$$

Proof. Using (1.1) we have

$$\tau_2(\gamma) = \nabla_T^3 T - kR(T,N)T$$
$$= (-3kk')T + (k'' + k^3 - k\tau^2)N + (2\tau k' + k\tau')B - kR(T,N)T.$$

From the vanishing of the normal components of $\tau_2(\gamma)$ we get

$$k'' + k^3 - k\tau^2 - kR(T,N,T,N) = 0, \qquad (4.2)$$
$$2\tau k' + k\tau' - kR(T,N,T,B) = 0.$$

Since $k \neq 0$ by the assumption that is non-geodesic. A direct computation using (3.4) yields

$$R(T,N,T,N) = 1 - 4B_1^2,$$

and

$$R(T,N,T,B) = 2N_1B_1$$

these, together with (4.2), complete the proof of the theorem.

5. EXAMPLE OF BIMINIMAL SURFACES IN THE LORENTZIAN HEISENBERG GROUP Heis³ (Heis³ LORENTZIAN HEISENBERG GRUBUNDA BİMİNİMAL YÜZEY ÖRNEKLERİ)

Let $\pi : Heis^3 \to \mathbb{R}^2$ be the projection $(x,y,z) \to (x,y)$. At a point $p = (x,y,z) \in Heis^3$ the vertical space of the submersion π is $V_p = Ker\{d\pi_p\} = span\{e_1\}$ and the horizontal space is $H_p = span\{e_2, e_3\}$. We have that the non zero covariant derivatives of the left invariant vector fields are:

$$\nabla_{e_1} e_1 = 0,\ \nabla_{e_1} e_2 = e_3, \nabla_{e_1} e_3 = e_2,$$
$$\nabla_{e_2} e_1 = e_3, \nabla_{e_2} e_2 = 0,\ \nabla_{e_2} e_3 = e_1,$$
$$\nabla_{e_3} e_1 = e_2, \nabla_{e_3} e_2 = -e_1, \nabla_{e_3} e_3 = 0.$$

Now let $\gamma(t) = (x(t), y(t))$ be a curve in $\mathbb{R}^2$ parametrized by arc length with signed curvature k and consider the flat cylinder $S = \pi^{-1}(\gamma)$ in $Heis^3$. Since the left invariant vector fields are orthonormal, the vector fields

$$E_1 = y' e_2 + x' e_3,\ E_2 = e_1 \qquad (5.1)$$

give an orthonormal frame tangent to S and

$$N = y' e_3 + x' e_2 \qquad (5.2)$$

is a unit normal vector field of S in $Heis^3$.

We now that the second fundamental form B of the surface $S = \pi^{-1}(\gamma)$, which is given by:

$$B=\begin{pmatrix} <\nabla_{E_1}E_1,N> & <\nabla_{E_1}E_2,N> \\ <\nabla_{E_2}E_1,N> & <\nabla_{E_2}E_2,N> \end{pmatrix}=\begin{pmatrix} k & -1 \\ -1 & 0 \end{pmatrix}. \tag{5.3}$$

From the expression of B we see that $H=\mathrm{trace}(B)/2=k/2$ and that $\|B\|^2=k^2+2$.

To write down the biminimality condition for S, we need to compute Ricci(N). For this, let us first recall that the non-zero components of the Riemann tensor of $Heis^3$ with respect to the left invariant vector fields are:

$$R_{1212}=R(e_1,e_2,e_1,e_2)=1,$$
$$R_{1313}=R(e_1,e_3,e_1,e_3)=-1,$$
$$R_{2323}=R(e_2,e_3,e_2,e_3)=3.$$

Then

$$\begin{aligned}\mathrm{Ricci}(N)&=R(E_1,N,E_1,N)+R(E_2,N,E_2,N)\\&=((x')^2-(y')^2)^2R_{2323}+(x')^2R_{1212}+(y')^2R_{1313}\\&=2.\end{aligned}$$

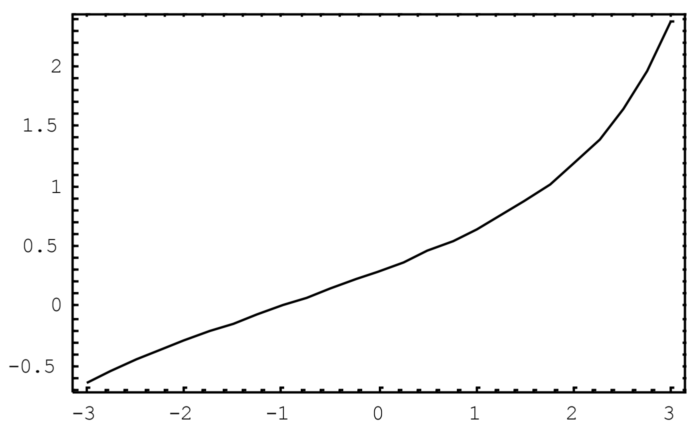
Thus, from (1.6), S is biminimal if and only if

$$\Delta H=(\|B\|^2-Ricci(N))H, \tag{5.4}$$

and using the computations, if and only if

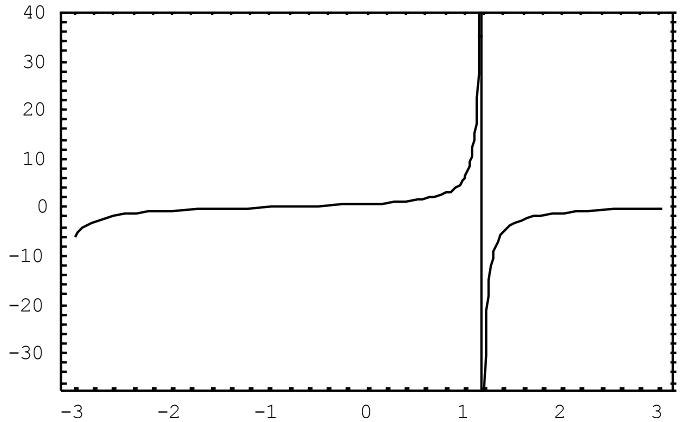
$$k''-k^3=0. \tag{5.5}$$

The picture of k can given as follows:

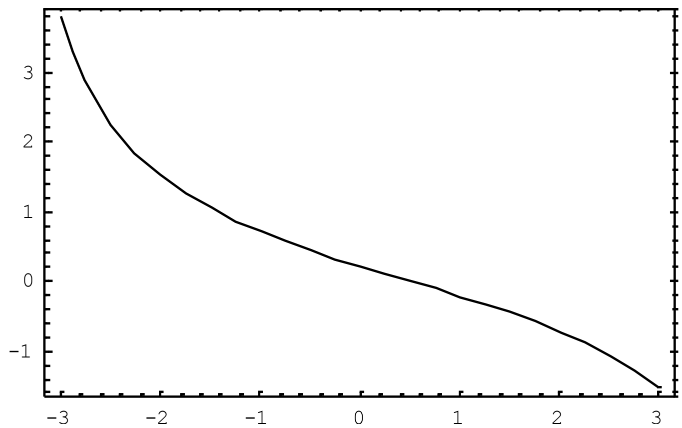


c1=c2=1, m=0.4

$$k(s)=\mp\frac{JacobiSN(\frac{0.5m\sqrt{2}(s+c2)}{\sqrt{m-c1}},m)\sqrt{-m-c1}}{c1}$$



c1=0.5, c2=1, m=0.4



c1=c2=0.5, m=0.3

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