ISSN:1306-3111
e-Journal of New World Sciences Academy
2012, Volume: 7, Number: 3, Article Number: 3A0054

NWSA-PHYSICAL SCIENCES
Received: April 2012
Accepted: July 2012
Series : 3A
ISSN : 1308-7304

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NULL BERTRAND CURVES OF THE AW(k)-TYPE IN MINKOWSKI 3-SPACE E_1^3

ABSTRACT

In this paper, we give curvature conditions of AW(k)-type (1≤k≤3) Cartan framed null Bertrand curves in Minkowski 3-space E_1^3. We obtain that, a null Bertrand curve which is not null geodesic is of AW(1) type if and only if its curvature is equal to its torsion and a null Bertrand curve and its Bertrand mate are of AW(2) and AW(3) type.

Keywords: AW (k)-Type Curve, Null Bertrand Curve

E_1^3 MINKOWSKI 3-UZAYINDA AW(k)-TİPİNDE NULL BERTRAND EĞRİLER

ÖZET

Bu çalışmada, Minkowski 3-uzayında AW(k) tipinde Cartan çatılı null Bertrand eğrileri için eğrilik şartlarını verdik. Bir null Bertrand eğrinin AW (1) tipinde olması için gerek ve yeter şartın eğriliğinin burulmasına eşit olmasıdır gerçeğini elde ettik. Ayrıca bir null Bertrand eğri ve onun Bertrand çiftinin AW(2) ve AW(3) tipinde olduklarını elde ettik.

Anahtar Kelimeler: AW (k) Tipinde Eğri, Null Bertrand Eğri

1. INTRODUCTION (GİRİŞ)

The geometry of null manifolds in spacetime has played an important role in the development of general relativity and electromagnetism. It is necessary to understand the causal structure of spacetimes, black holes, asymptotically flat systems and gravitational waves. Null curves are 1-dimensional degenere manifolds. In this sense, the null curves in Minkowski space have been studied by both mathematicians and physicists (see [5, 9 and 10]). Null curves have many properties very different from spacelike or timelike curves.

From the differential geometric point of view, the study of null curves has its own interest. Many interesting results on null curves have been obtained by many mathematicians (see [4 and 6]). When we study these curves, some difficulties arise because the arclength vanishes, so that it is not possible to normalize the tangent vector in the usual way. Thus, a new parameter called the pseudoarc which normalizes the derivate of the tangent vector is introduced. Many authors defined a Frenet frame with the minimum number of curvature functions (which call the Cartan frame) for a null curve in an n-dimensional Lorentzion space form.

In this paper, we will give null Bertrand curves of AW(k)-type, $1 \leq k \leq 3$. Many studies on curves of AW(k)-type have been done by many mathematicians (see [1, 2, 3, 7 and 8]). In the literature, there is no null Bertrand curves related with curves of AW(k)-type. Therefore, in this paper we have done a study on null Bertrand curves of AW(k)-type. The purpose of this paper is to implement the results which were given [1] to null curves of AW(k)-type in Minkowski space.

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

Let E_1^3 be a Minkowski 3-space with natural Lorentz metric

$$\langle .,\ . \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

In terms of natural coordinates.

The vector product operation of E_1^3 is defined by

$$x \times y = (x_3 y_2 - x_2 y_3 \ , \ x_3 y_1 - x_1 y_3 \ , \ x_1 y_2 - x_2 y_1)$$

for $x=(x_1, x_2, x_3)$, $y=(y_1, y_2, y_3) \in E_1^3$.

A tangent vector v of E_1^3 is said to be spacelike if $\langle v,v\rangle > 0$ or $v=0$, timelike if $\langle v,v\rangle < 0$, lightlike or null if $\langle v,v\rangle = 0$ and $v \neq 0$.

A parametrized curve $\alpha = \alpha(s)$ in Minkowski 3-space E_1^3 is called a null curve if its tangent vector field is null,i.e., $\langle \alpha', \alpha' \rangle = 0$, $\alpha' \neq 0$.

Definition 2.1. A curve $\alpha = \alpha(s)$ in Minkowski 3-space E_1^3 said to be a null Frenet curve(or a Cartan framed null curve) if it admits a Frenet frame field { ℓ, n, u } such that

$$\begin{aligned} \ell' &= \kappa u \\ n' &= -\tau u \\ u' &= -\tau \ell + \kappa n \end{aligned} \qquad (2.1)$$

with $\ell = d\alpha / ds$, $\langle \ell, \ell \rangle = \langle n,n \rangle = 0$, $\langle \ell, n \rangle = -1$ and u is defined by $u = \ell \times n$. The functions κ and τ are called the curvature and torsion of α, respectively.

We call the vector fields ℓ, n and u a tangent vector field, a binormal vector field and a (principal) normal vector field of α, respectively, [6].

Definition 2.2. A null curve with respect to a Cartan frame with τ = 0 is called a generalized null cubic.

Definition 2.3. A null curve with respect to a Cartan frame with κ= 0 is called a null geodesic, [5].

The null curve α is called a null Frenet curve of oscillating order 3 if its derivates $\alpha'(s), \alpha''(s), \alpha'''(s)$ are linearly independent and $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha^{(4)}(s)$ are no longer linearly independent for all s∈I. To each null Frenet curve of order 3 one can associate a Cartan frame $\{\ell, n, u\}$ along α.

Proposition 2.4. Let α be a Cartan framed curve of E_1^3 of osculating order 3, then we have

$$\alpha'(s)=\ell(s)$$
$$\alpha''(s)=\kappa(s)u(s) \quad (2.2)$$
$$\alpha'''(s)=-\kappa(s)\tau(s)\ell(s)+\kappa(s)\tau(s)n(s)+\kappa'(s)u(s)$$
$$\alpha^{(4)}(s)=[-\kappa(s)\tau(s)-2\kappa'(s)\tau(s)]\ell(s) +[-\kappa'(s)\kappa(s)+\kappa'(s)\tau(s)+\kappa(s)\tau'(s)]n(s) +[\kappa''(s)+\kappa^2(s)\tau(s)-\kappa(s)\tau^2(s)]u(s).$$

Notation: Let us write

$$N_1(s) = \kappa u \quad (2.3)$$
$$N_2(s) = \kappa\tau n + \kappa' u \quad (2.4)$$
$$N_3(s) = (-\kappa'\kappa+\kappa'\tau+\kappa\tau')n + (\kappa''+\kappa^2\tau-\kappa\tau^2)u \quad (2.5)$$

Corollary2.5. $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha^{(4)}(s)$ are linearly dependent if and only if $N_1(s), N_2(s), N_3(s)$ are linearly dependent.

3. NULL BERTRAND CURVES OF AW(k)-TYPE (AW(k)- TİPİNDE NULL BERTRAND EĞRİLER)

In this section we consider Cartan framed null curve and null Bertrand curves of AW(k)-type.

Definition 3.1. Cartan framed null curves are

i) of type AW(1) if they satify $N_3(p) = 0$

ii) of type AW(2) if they satify

$$||N_2(s)||^2 N_3(s)=\langle N_3(s),N_2(s)\rangle N_2(s) \quad (3.1)$$

iii) of type AW(3) if they satify

$$||N_1(s)||^2 N_3(s)=\langle N_3(s),N_1(s)\rangle N_1(s) \quad (3.2)$$

Proposition 3.2. Let α be a Cartan framed null curve of order 3. Then α is of type AW(1) if and only if

$$-\kappa'\kappa+\kappa'\tau+\kappa\tau'=0 \quad (3.3)$$

and

$$\kappa''+\kappa^2\tau-\kappa\tau^2=0 \quad (3.4)$$

Proof. Let α be a Cartan framed null curve of type AW(1). From Definition 3.1.i), $N_3(P)$ =0. Then from (2.5) equality , we have

$$(-\kappa'\kappa+\kappa'\tau+\kappa\tau')n+(\kappa''+\kappa^2\tau-\kappa\tau^2)u=0.$$

Furthermore, since n and u are linearly independent , one can obtain (3.3) and (3.4). Similarly, the converse statement can be proved. The proof is completed.

Proposition 3.3. Let α be a Cartan framed null curve of order 3. Then α is of type AW(2) if and only if

$$-(\kappa')^3\kappa+(\kappa')^3\tau+\kappa(\kappa')^2\tau'=\kappa\tau\kappa'\kappa''+\kappa^3\kappa'\tau^2-\kappa^2\kappa'\tau^3 \quad (3.5)$$

Proof. If α is type of AW(2),(3.1)holds on α. Substituting (2.4) and (2.5) into (3.1), one can obtain(3.5). The converse statement is trivial. This completes the proof.

Proposition 3.4. Let α be a Cartan framed null curve of order 3. Then α is of type AW(3) if and only if

$$-\kappa'\kappa^3+\kappa'\kappa^2\tau+\kappa^3\tau'=0. \quad (3.6)$$

Proof. Since α is of type AW(3), (3.2) holds on α. So substituting (2.3) and (2.5) into (3.2), we have (3.6). The converse statement is trivial. Hence our proposition is proved.

Definition 3.5. Let α and $\overline{\alpha}$ be two Cartan framed null curves in E_1^3. Then a pair of curves (α, $\overline{\alpha}$) is said to be a null Bertrand pair if u and $\overline{u}$ are linearly dependent.

The curve $\overline{\alpha}$ is called a null Bertrand mate of α and vice verca. A Cartan framed null curve is said to be a null Bertrand curve if it admits a Bertrand mate, [4].

Teorem 3.6. Let α be a Cartan Framed null curve. Then α is a null Bertrand curve if and only if α is a null geodesic or a Cartan framed null curve with constant torsion τ ($\neq 0$), [4].

Corollary 3.7. Let (α, $\overline{\alpha}$) be a Bertrand pair of Cartan framed null curves which are not geodesics. Then their curvature functions satisfy the following relations:

$$\overline{\kappa}\,\kappa = \text{const.} > 0 \quad \text{and} \quad \overline{\tau} = \tau = \text{const.} \ (= 1/r),$$

where r is nonzero constant.

Theorem 3.6. implies that every Cartan framed proper null helix admits a null Bertrand mate. Moreover, by Corollary3.7 the null Bertrand mate is also a proper null helix, [4].

The followings are the main results of this paper:

Teorem 3.8. Let α be a null Bertrand curve which is not null geodesic. Then α is of AW(1) type if and only if $\kappa = \tau$ on α.

Proof. Let α be a null Bertrand curve. Then κ=0 or τ=const.($\neq$0) on α. In this case the equations (3.3) and (3.4) hold on α. From Proposition3.2, α is a null Bertrand curve of AW(1) type.

In case τ=const., from Corollary3.7 we have κ=const., By using equation (3.4) we obtain $\kappa = \tau$. Thus the equations (3.3) and (3.4) hold on α. This means that α is of AW(1) type.

Teorem 3.9. Let α be a null Bertrand curve. Then α and its Bertrand mate $\overline{\alpha}$ are of AW(2)type.

Proof. If α is null geodesic , the proof is clear. If α is a null Bertrand curve which is not geodesic. Considering Corollary3.7, κ and τ are nonzero constant on α. Since the Bertrand mate of α is a null helix, The equation (3.5) holds on the Bertrand mate of α. Thus the proof is completed.

Teorem 3.10. Let α be a null Bertrand curve. Then α and its Bertrand mate $\overline{\alpha}$ are of AW(3)type.

Proof. Since α is a null Bertrand curve. Then α is a null geodesic or a Cartan framed null curve with nonzero constant torsion τ. If κ=0, Equation (3.6) holds on α. If τ is nonzero constant, then α is a null helix with nonzero constant curvature. In this case, the equation (3.6) holds on α. From Proposition3.4, α is of AW(3) type. Considering the Corollary 3.7, it is trivial that, $\overline{\alpha}$ is of AW(3) type. This completes the proof.

Example 3.11. Null geodesics can be given an example for Theorem 3.8.

Example 3.12. Let α be a parametrized null curve defined by (see Figure 1)

$$\alpha(s) = (\frac{1}{2}\sinh(2s), \frac{1}{2}\cosh(2s), s).$$

The α is framed by a Cartan frame F = (ℓ,n, u):

$$\ell(s) = \alpha'(s) = (\cosh(2s), \sinh(2s), 1)$$

$n(s)=(\frac{1}{2}\cosh(2s), \frac{1}{2}\sinh(2s), -\frac{1}{2})$

$u(s)=(\sinh(2s), \cosh(2s), 0)$.

The curvature functions of α with respect to F are $\kappa=2$, $\tau=-1$. Define $\bar{s}=(-2/\mu)s$, $s \in R$. Then the curve

$$\bar{\alpha}(\bar{s}) = \alpha(s)-u(s) = -\frac{1}{2}(\sinh(-\mu\bar{s}), \cosh(-\mu\bar{s}), \mu\bar{s})$$

gives a one-parameter family of Bertrand mates of α framed by

$$\bar{l}(\bar{s})=\mu n(s), \quad \bar{n}(\bar{s})=\mu^{-1}\ell(s), \quad \bar{u}(\bar{s})=-u(s),$$

where μ is nonzero constant. The curvature functions of $\bar{\alpha}$ are

$$\bar{\kappa}=\frac{\mu^2}{2}, \quad \bar{\tau}=-1.$$

Thus this null Bertrand curve α and its Bertrand mate $\bar{\alpha}$ are example for Theorem 3.9 and Theorem 3.10.

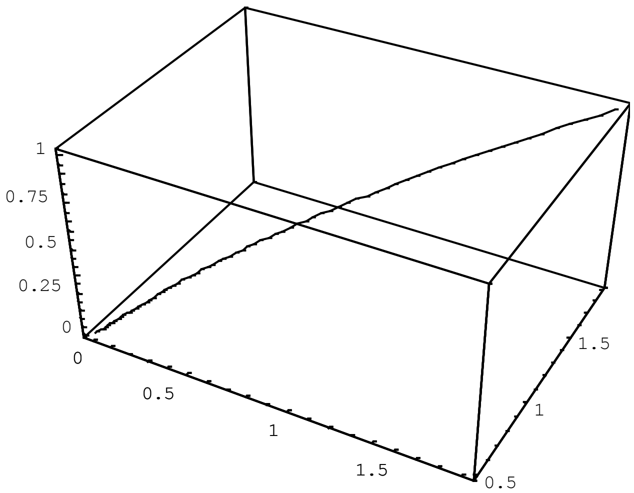

Figure 1. The null curve of AW(2) and AW(3) type
Şekil 1. AW (2) ve AW ve boş bir eğri (3) tipi

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