ISSN:1306-3111
e-Journal of New World Sciences Academy
2011, Volume: 6, Number: 2, Article Number: 3A0034

PHYSICAL SCIENCES
Received: November 2010
Accepted: February 2011
Series : 3A
ISSN : 1308-7304

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ON ORDER STATISTICS OF CONTINUOUS RANDOM VARIABLES

ABSTRACT

In this study, the *df* and *pdf* of the *r*th order statistic arising from *innid* continuous random variables are expressed. Then, the results related to distributions of minimum and maximum of *innid* continuous random variables are given.

Keywords: Order Statistics, Permanent, Continuous Random Variable, Distribution Function, Probability Density Function

SÜREKLİ TESADÜFİ DEĞİŞKENLERİN SIRALI İSTATİSTİKLERİ ÜZERİNE

ÖZET

Bu çalışmada, *innid* sürekli tesadüfi değişkenlerinin *r*-inci sıralı istatistiğinin dağılım ve olasılık yoğunluk fonksiyonları ifade edilmiştir. Sonra, *innid* sürekli tesadüfi değişkenlerin minimum ve maksimumunun dağılımları ile ilgili sonuçlar verilmiştir.

Anahtar Kelimeler: Sıralı İstatistikler, Permanent, Sürekli Tesadüfi Değişkenler, Dağılım Fonksiyonu, Olasılık Yoğunluk Fonksiyonu

1. INTRODUCTION (GİRİŞ)

Several identities and recurrence relations for probability density function (*pdf*) and distribution function (*df*) of order statistics of independent and identically distributed (*iid*) random variables were established by numerous authors including Arnold et al. [1], Balasubramanian and Beg [3], David [13], and Reiss [18]. Furthermore, Arnold et al. [1], David [13], Gan and Bain [14], and Khatri [17] obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Corley [11] defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller [15]. Guilbaud [16] expressed the probability of the functions of independent but not necessarily identically distributed (*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West [9]. In addition, Vaughan and Venables [19] derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. Balakrishnan [2], and Bapat and Beg [7] obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents.

Using multinomial arguments, the *pdf* of $X_{r:n+1}$ ($1 \le r \le n+1$) was obtained by Childs and Balakrishnan [10] by adding another independent random variable to the original *n* variables $X_1, X_2, ..., X_n$. Also, Balasubramanian et al. [6] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators.

In a paper published in 1991, Beg [8] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer et al. [12] derived the expressions for the distribution and density functions by Ryser’s method and the distribution of maxima and minima based on permanents. In the first of two papers, Balasubramanian et al. [4] obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, ..., X_n\}$ where X_i’s are *innid* random variables. Later, Balasubramanian et al. [5] generalized their previous results [4] to the case of the joint distribution function of several order statistics.

From now on, the subscripts and superscripts are defined in the first place in which they are used and these definitions will be valid unless they are redefined.

If $\mathbf{a}_1, \mathbf{a}_2, \ldots$ are defined as column vectors, then the matrix obtained by taking m_1 copies of $\mathbf{a}_1$, m_2 copies of $\mathbf{a}_2$, … can be denoted as $[\underset{m_1}{\mathbf{a}_1} \ \underset{m_2}{\mathbf{a}_2} \ \ldots]$ and *per*A denotes the permanent of a square matrix A, which is defined as similar to determinants except that all terms in the expansion have a positive sign.

Let $X_1, X_2, ..., X_n$ be *innid* continuous random variables and $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be the order statistics obtained by arranging the *n* X_i's in increasing order of magnitude.

Let F_i and f_i be *df* and *pdf* of X_i (*i=1, 2,…, n*), respectively.

The paper is organized as follows. In section 3, we give the theorems concerning *df* and *pdf* of order statistics if *innid* continuous random variables. In the last section, some results related to *df* and *pdf* will be given.

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In general, the distribution theory for order statistics is complicated when the random variables are *innid*. In this study, the distributions of the *r*th order statistics from *innid* random variables are obtained easily by using permanent.

3. THEOREMS FOR DISTRIBUTION AND PROBABILITY DENSITY FUNCTIONS (DAĞILIM VE OLASILIK YOĞUNLUK FONKSİYONLARI İÇİN TEOREMLER)

In this section, the theorems related to *df* and *pdf* of $X_{r:n}$ will be given. We will now express the following theorem for the *df* of *r*th order statistic of *innid* continuous random variables.

Theorem 1.

$$F_{r:n}(x) = \sum_{m=r}^{n} \frac{1}{m!(n-m)!} \sum_{t=m}^{n} (-1)^{n-t} \binom{n-m}{t-m} \sum_{n_s=n-t+m} (t-m)!\, per[\underset{n-t+m}{\mathrm{F}(x)}][s/.] , \tag{1}$$

where $\mathrm{F}(x) = (F_1(x), F_2(x), ..., F_n(x))'$ is column vector, $x \in R$, s is a non-empty subset of the integers {*1, 2,…, n*} with $n_s \ge 1$ elements and $\mathrm{A}[s/.)$ is the matrix obtained from A by taking rows whose indices are in s.

Proof. It can be written

$$F_{r:n}(x) = P\{X_{r:n} \le x\}. \tag{2}$$

(2) can be expressed as

$$F_{r:n}(x) = \sum_{m=r}^{n} \frac{1}{m!(n-m)!} per\mathrm{A} , \tag{3}$$

where $\mathrm{A} = [\underset{m}{\mathrm{F}(x)} \quad \underset{n-m}{1-\mathrm{F}(x)}]$ is matrix and $1-\mathrm{F}(x) = (1-F_1(x), 1-F_2(x), ..., 1-F_n(x))'$.

Using properties of permanent, we can write

$$per\mathrm{A} = per[\underset{m}{\mathrm{F}(x)} \quad \underset{n-m}{1-\mathrm{F}(x)}]$$

$$= \sum_{t=0}^{n-m} (-1)^{n-m-t} \binom{n-m}{t} per[\underset{n-t}{\mathrm{F}(x)} \quad \underset{t}{1}\ ]$$

$$= \sum_{t=0}^{n-m} (-1)^{n-m-t} \binom{n-m}{t} \sum_{n_s=n-t} t!\, per[\underset{n-t}{\mathrm{F}(x)}][s/.)$$

$$= \sum_{t=m}^{n} (-1)^{n-t} \binom{n-m}{t-m} \sum_{n_s=n-t+m} (t-m)!\, per[\underset{n-t+m}{\mathrm{F}(x)}][s/.) \quad , \tag{4}$$

where $1 = (1,1,...,1)'$. Using (4) in (3), (1) is obtained.

We will now express the following theorem for the *pdf* of *r*th order statistic of *innid* continuous random variables.

Theorem 2.

$$f_{r:n}(x) = \frac{1}{(r-1)!(n-r)!}\sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}\sum_{n_{s}=n+r-t}(t-r)!\sum_{n_{\varsigma}=n+r-1-t} per[\underset{n+r-1-t}{\mathrm{F}(x)}][\varsigma/.)\, per[\underset{1}{\mathrm{f}(x)}][\varsigma'/.) \quad (5)$$

where $\mathrm{f}(x)=(f_1(x),f_2(x),\dots,f_n(x))'$, $s=\varsigma\cup\varsigma'$, $\varsigma\cap\varsigma'=\phi$ and $n_{\varsigma'}=1$.

Proof. Consider

$$P\{x<X_{r:n}\le x+\delta x\}. \quad (6)$$

Dividing (6) by δx and then letting δx tend to zero, we obtain

$$f_{r:n}(x)=D\,per\mathrm{B}, \quad (7)$$

where $\mathrm{B}=[\underset{r-1}{\mathrm{F}(x)}\ \ \underset{1}{\mathrm{f}(x)}\ \ \underset{n-r}{1-\mathrm{F}(x)}]$ is matrix. Using properties of permanent, we can write

$$\begin{aligned}
per\mathrm{B} &= per[\underset{r-1}{\mathrm{F}(x)}\ \ \underset{1}{\mathrm{f}(x)}\ \ \underset{n-r}{1-\mathrm{F}(x)}] \\
&= \sum_{t=0}^{n-r}(-1)^{n-r-t}\binom{n-r}{t} per[\underset{n-1-t}{\mathrm{F}(x)}\ \ \underset{1}{\mathrm{f}(x)}\ \ \underset{t}{1}] \\
&= \sum_{t=0}^{n-r}(-1)^{n-r-t}\binom{n-r}{t}\sum_{n_{s}=n-t} t!\, per[\underset{n-1-t}{\mathrm{F}(x)}\ \ \underset{1}{\mathrm{f}(x)}][s/.) \\
&= \sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}\sum_{n_{s}=n+r-t}(t-r)!\, per[\underset{n+r-1-t}{\mathrm{F}(x)}\ \ \underset{1}{\mathrm{f}(x)}][s/.) \\
&= \sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}\sum_{n_{s}=n+r-t}(t-r)!\sum_{n_{\varsigma}=n+r-1-t} per[\underset{n+r-1-t}{\mathrm{F}(x)}][\varsigma/.)\, per[\underset{1}{\mathrm{f}(x)}][\varsigma'/.). \quad (8)
\end{aligned}$$

Using (8) in (7), (5) is obtained.

4. RESULTS (SONUÇLAR)

In this section, the results related to *df* and *pdf* of $X_{r:n}$ will be given. We will now express the following result for *df* of minimum order statistic.

Result 1.

$$F_{1:n}(x)=1-\frac{1}{n!}\sum_{t=0}^{n}(-1)^{n-t}\binom{n}{t}\sum_{n_s=n-t} t!\, per[\underset{n-t}{\mathrm{F}(x)}][s/.). \quad (9)$$

Proof. In (1), if $r=1$, (9) is obtained.

We will express the following result for *df* of maximum order statistic.

Result 2.

$$F_{n:n}(x)=\frac{1}{n!}per[\underset{n}{\mathrm{F}(x)}]. \quad (10)$$

Proof. In (1), if $r=n$, (10) is obtained.

In the following result, we will express *pdf* of minimum order statistic.

Result 3.

$$f_{1:n}(x)=\frac{1}{(n-1)!}\sum_{t=1}^{n}(-1)^{n-t}\binom{n-1}{t-1}\sum_{n_{s}=n+1-t}(t-1)!\sum_{n_{\varsigma}=n-t} per[\underset{n-t}{\mathrm{F}(x)}][\varsigma/.)\, per[\underset{1}{\mathrm{f}(x)}][\varsigma'/.).$$

(11)

Proof. In (5), if $r=1$, (11) is obtained.

We will express the following result for *pdf* of maximum order statistic.

Result 4.

$$f_{n:n}(x) = \frac{1}{(n-1)!} \sum_{n_\varsigma = n-1} \underset{n-1}{per[\mathrm{F}(x)][\varsigma/.)}\ \underset{1}{per[\mathrm{f}(x)][\varsigma'/.)} \quad . \tag{12}$$

Proof. In (5), if $r = n$, (12) is obtained.

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