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BIVARIATE FARLIE-GUMBEL-MORGENSTERN DISTRIBUTIONS

ABSTRACT

In this study, the bivariate Farlie-Gumbel-Morgenstern (FGM) distributions by introducing association parameter is considered. For the bivariate FGM distributions, the admissible range of the association parameter is found. Also, positive quadrant dependence property is shown. Furthermore, the Pearson correlation coefficient for uniform marginals is calculated. The application in financial data is demonstrated.

Keywords: Bivariate FGM Distributions, Association Parameter, Admissible Range, Positive Quadrant Dependence, Pearson Correlation Coefficient

İKİ DEĞİŞKENLİ FARLIE-GUMBEL-MORGENSTERN DAĞILIMLARI

ÖZET

Bu çalışmada, birliktelik parametresi ile sunulan iki değişkenli Farlie-Gumbel-Morgenstern (FGM) dağılımları incelenmiştir. İki değişkenli FGM dağılımları için, birliktelik parametresinin kabul edilebilir sınırı bulunmuştur. Ayrıca, pozitif kadran bağımlılık özelliği gösterilmiştir. Bundan başka, uniform marjinaller için Pearson korelasyon katsayısı hesaplanmıştır. Finansal verilerde uygulama örnekle açıklanmıştır.

Anahtar Kelimeler: İki Değişkenli FGM Dağılımları, Birliktelik Parametresi, Kabul Edilebilir Sınır, Pozitif Kadran Bağımlılık, Pearson Korelasyon Katsayısı

1. INTRODUCTION (GİRİŞ)

Let (X,Y) be a bivariate absolutely continuous random variable, formally defined by the joint distribution function (d.f.)

$$F_{X,Y}(x,y)=F(x)G(y)\{1+\alpha A(F(x))B(G(y))\} \tag{1}$$

where $A(x)\to 0$ and $B(x)\to 0$ as $x\to 1$ and the 'kernels' $A(x)$, $B(x)$ satisfy certain regularity conditions ensuring that equation (1) is a d.f. with absolutely continuous marginals $F(x)$ and $G(y)$. Bivariate distributions with d.f. (1) are usually referred to as FGM distributions. This model was originally introduced by Morgenstern [1] for $A(x)=1-x$, $B(y)=1-y$ and investigated by Gumbel [2] for exponential marginals. Subsequent generalization to form (1) is due to Farlie [3], Johnson and Kotz [4]. The admissible range of association parameter α for the distribution with $A(x)=1-x$, $B(y)=1-y$ is $-1\le\alpha\le 1$ and the Pearson correlation coefficient ρ between X and Y can never exceed $1/3$.

The multivariate case has been studied by Johnson and Kotz [4 and 5] among others. Huang and Kotz [6] used successive iterations in the original FGM distribution to increase the correlation between components. For example, the successive iterations with uniform marginals give:

$$H_{\alpha,\beta}(x,y)=xy\{1+\alpha(1-x)(1-y)+\beta xy(1-x)(1-y)\}\ ,\ 0\le x,y\le 1\,. \tag{2}$$

For this distribution, the correlation coefficient is $\rho=\dfrac{\alpha}{3}+\dfrac{\beta}{12}$. In this case, the admissible range of α is as above $-1\le\alpha\le 1$, but the range of β depends on α. The maximal correlation coefficient attained for this family is $\rho_{\max}=0.434$ versus $\rho_{\max}=1/3$ achieved for $\alpha=1$ in the original FGM version. Huang and Kotz [7] considered a polynomial-type single parameter extensions of FGM (with uniform marginals)

$$H_{\alpha}(x,y)=xy\{1+\alpha(1-x^{p})(1-y^{p})\}\ ,\ p\ge 1,\ 0\le x,y\le 1 \tag{3}$$

and

$$H_{\alpha}^{1}(x,y)=xy\{1+\alpha(1-x)^{q}(1-y)^{q}\}\ ,\ q>1,\ 0\le x,y\le 1\,. \tag{4}$$

For equation (3), the admissible range of α is:

$$-\left(\max\{1,p\}\right)^{-2}\le\alpha\le p^{-1} \tag{5}$$

and the range for correlation coefficient is:

$$-3(p+2)^{-2}\min\{1,p^{2}\}\le\rho\le\frac{3p}{(p+2)^{2}}\,. \tag{6}$$

Similarly for (4), the bound

$$-1\le\alpha\le\left(\frac{q+1}{q-1}\right)^{q-1} \tag{7}$$

on α is translated into

$$-12((q+1)(q+2))^{-2}\le\rho\le 12(q-1)^{1-q}(q+1)^{q-3}(q+2)^{-2}\,. \tag{8}$$

The maximal positive correlation for equal (3) $(\rho=3/8)$ is attained for $p=2$, an improvement over the case $p=1$ for which $\rho=1/3$.

Bairamov and Kotz [8] provided several theorems characterizing symmetry and dependence properties of FGM distributions. They also proposed a modification by introducing additional parameter α:

$$F_{p,q,\alpha}(x,y)=xy\{1+\alpha(1-x^{p})^{q}(1-y^{p})^{q}\}\ ,\ p\ge 1,\ q>1,\ 0\le x,y\le 1\,. \tag{9}$$

For equation (9) the admissible range of α is:

$$-\min\left\{1, \frac{1}{p^2}\left(\frac{1+pq}{p(q-1)}\right)^{2(q-1)}\right\} \le \alpha \le \frac{1}{p}\left(\frac{1+pq}{p(q-1)}\right)^{q-1} . \qquad (10)$$

The maximal and minimal values of ρ within this range are

$$-12t^2(q,p)\min\left\{1, \frac{1}{p^2}\left(\frac{1+pq}{p(q-1)}\right)^{2(q-1)}\right\} \le \rho \le 12t^2(q,p)\frac{1}{p}\left(\frac{1+pq}{p(q-1)}\right)^{q-1} \qquad (11)$$

where $t(x,y) = \frac{\Gamma(x+1)\Gamma(2/y)}{y\Gamma(x+1+2/y)}$.

In this case, the maximal strongest positive correlation is $\rho_{\max} = 0.5021$ attained at $q = 1.496$ and $p = 3$. Hence, the extension (9) can achieve correlation greater than $1/2$ compared with the classical FGM where the correlation cannot be greater than $1/3$. Such an extension of the range of the correlation is clearly useful in practical applications. If two components in a system have correlation greater than 0.5, this distribution is useful to model systems data.

Recently, Lai and Xie [9], using the uniform representation of the FGM bivariate distribution, introduced and studied continuous bivariate distributions possessing a positive quadrant dependence (PQD) property with the association parameter contained between 0 and 1. Recall that random variables X and Y are called positively quadrant dependent (PQD) if the inequality

$$P\{X \le x, Y \le y\} \ge P\{X \le x\}P\{Y \le y\} \text{ for all } x \text{ and } y \qquad (12)$$

holds. Let $F(x,y)$ denote the joint d.f. of (X,Y) having marginal d.f.'s $F_X(x)$ and $F_Y(y)$. As pointed out by Lai and Xie [9], for PQD bivariate distributions the joint d.f. may be written in the form

$$F(x,y) = F_X(x)F_Y(y) + w(x,y) \text{ for all } x \text{ and } y \qquad (13)$$

with non-negative $w(x,y)$ satisfying certain regularity conditions ensuring that $F(x,y)$ is a d.f.. Lai and Xie [9] considered a bivariate function

$$C(u,v) = uv + \alpha u^b v^b (1-u)^a (1-v)^a \quad , \quad a,b \ge 1, \quad 0 \le u,v \le 1 \qquad (14)$$

and proved that $C(u,v)$ is bivariate PQD copula for $0 \le \alpha \le 1$. Recall that a bivariate copula is a bivariate d.f. with uniform marginals.

Note that utilizing equations (5) and (7), one can construct other PQD copulas

$$C(u,v) = uv\{1 + \alpha(1-u^p)(1-v^p)\} \quad , \quad p \ge 1, \quad 0 \le \alpha \le p^{-1}, \quad 0 \le u,v \le 1 \qquad (15)$$

$$C(u,v) = uv\{1 + \alpha(1-u)^q(1-v)^q\} \quad , \quad q > 1, \quad 0 \le \alpha \le \left(\frac{q+1}{q-1}\right)^{q-1}, \quad 0 \le u,v \le 1$$

(16)

and by considering (10), the PQD copula

$$C(u,v) = uv\{1 + \alpha(1-u^p)^q(1-v^p)^q\} \quad , \quad p \ge 1, \quad q > 1, \quad 0 \le u,v \le 1 \qquad (17)$$

is obtained with

$$0 \le \alpha \le \frac{1}{p}\left(\frac{1+pq}{p(q-1)}\right)^{q-1} . \qquad (18)$$

It is known that most bivariate distributions in reliability theory are PQD, see for example Hutchinson and Lai [10]. If we calculate the reliability of a series system assuming independence of components when in fact they are PQD, we will overestimate the system reliability.

Bairamov and Kotz studied on a new family of PQD bivariate distributions [11].

Bairamov, Kotz and Bekçi [12] introduced a new generalized FGM distributions. The generalization of FGM distribution is given by:

$$F_{p_1,p_2,q_1,q_2,\alpha}(x,y)=xy\left\{1+\alpha\left(1-x^{p_1}\right)^{q_1}\left(1-y^{p_2}\right)^{q_2}\right\}\ ,\quad p_1,p_2\geq 1,\quad q_1,q_2>1,\quad 0\leq x,y\leq 1\,. \tag{19}$$

They showed that equation (19) is a copula for α satisfying

$$-\min\left\{1,\frac{1}{p_1p_2}\left(\frac{1+p_1q_1}{p_1(q_1-1)}\right)^{q_1-1}\left(\frac{1+p_2q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\}\leq\alpha\leq\min\left\{\frac{1}{p_1}\left(\frac{1+p_1q_1}{p_1(q_1-1)}\right)^{q_1-1},\frac{1}{p_2}\left(\frac{1+p_2q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\} \tag{20}$$

and possesses PQD property for an α satisfying

$$0\leq\alpha\leq\min\left\{\frac{1}{p_1}\left(\frac{1+p_1q_1}{p_1(q_1-1)}\right)^{q_1-1},\frac{1}{p_2}\left(\frac{1+p_2q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\}. \tag{21}$$

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this study, the bivariate FGM distributions are investigated. Admissible range of association parameter α for the bivariate FGM distribution is obtained. These distributions are extensively used in system analysis, reliability theory, survival analysis, finance and etc.. The bivariate FGM distributions can be used for financial data in Turkey.

3. ANALYTICAL STUDY (ANALİTİK ÇALIŞMA)

In the following theorem, the admissible range of association parameter α for the bivariate FGM distribution is obtained.

Theorem 1. Let (X,Y) be a bivariate absolutely continuous random variable, the bivariate joint d.f. is given by

$$F(x,y)=xy\left\{1+\alpha x^{q_1}y^{q_2}\left(1-x^{p_1}\right)\left(1-y^{p_2}\right)\right\}\ ,\quad p_1,p_2\geq 1,\quad q_1,q_2\geq 1,\quad 0\leq x,y\leq 1\,. \tag{22}$$

For equation (22), the admissible range of α is

$$-\min\left\{\frac{1}{p_1p_2},\frac{1}{c_1(p_1,q_1)c_2(p_2,q_2)}\right\}\leq\alpha\leq\min\left\{\frac{1}{p_1c_2(p_2,q_2)},\frac{1}{p_2c_1(p_1,q_1)}\right\} \tag{23}$$

where

$$c_1(p_1,q_1)=\left[\frac{q_1(q_1+1)}{(p_1+q_1)(p_1+q_1+1)}\right]^{q_1/p_1}\frac{p_1(q_1+1)}{p_1+q_1}, \tag{24}$$

$$c_2(p_2,q_2)=\left[\frac{q_2(q_2+1)}{(p_2+q_2)(p_2+q_2+1)}\right]^{q_2/p_2}\frac{p_2(q_2+1)}{p_2+q_2}. \tag{25}$$

If

$$0\leq\alpha\leq\min\left\{\frac{1}{p_1c_2(p_2,q_2)},\frac{1}{p_2c_1(p_1,q_1)}\right\} \tag{26}$$

then it possesses PQD property.

Proof. It is easily verified that the joint probability density function (p.d.f.) of equation (22) is

$$f(x,y)=1+\alpha x^{q_1}\left[q_1+1-(p_1+q_1+1)x^{p_1}\right]y^{q_2}\left[q_2+1-(p_2+q_2+1)y^{p_2}\right] \tag{27}$$

$$p_1,p_2\geq 1,\quad q_1,q_2\geq 1,\quad 0\leq x,y\leq 1\,.$$

The overall constraint on α is given by

$$\alpha x^{q_1}\left[q_1+1-(p_1+q_1+1)x^{p_1}\right]y^{q_2}\left[q_2+1-(p_2+q_2+1)y^{p_2}\right]\geq -1\,. \tag{28}$$

It is clear that $f(x,y)=1$ for all values of α on the lines

$$\tilde{x}=\left[\frac{q_1+1}{p_1+q_1+1}\right]^{1/p_1}\quad\text{and}\quad\tilde{y}=\left[\frac{q_2+1}{p_2+q_2+1}\right]^{1/p_2} \tag{29}$$

which mark the boundaries of the quadrants Q_1, Q_2, Q_3 and Q_4 which subdivide unit square into four parts (see, Figure 1.).

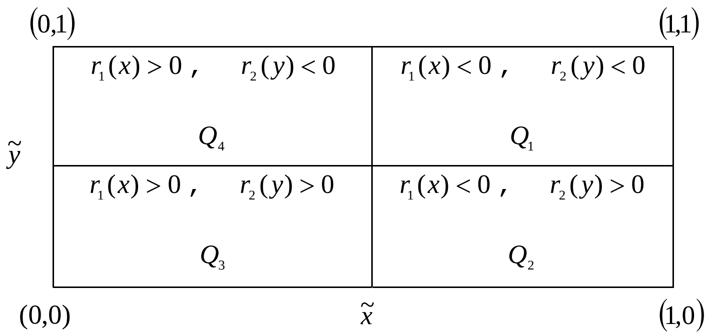

Figure 1. Schematic representation of the quadrants
(Şekil 1. Kadranların şematik gösterimi)

Consider the function $r_1(x)=x^{q_1}\left[q_1+1-(p_1+q_1+1)x^{p_1}\right]$. It is easy to verify that the solutions of

$$r_1'(x)=x^{q_1-1}\left[q_1(q_1+1)-(p_1+q_1)(p_1+q_1+1)x^{p_1}\right]=0 \tag{30}$$

i.e., the extreme point of $r_1(x)$ in $0\le x\le 1$ is

$$x_*=\left[\frac{q_1(q_1+1)}{(p_1+q_1)(p_1+q_1+1)}\right]^{1/p_1}. \tag{31}$$

Further analysis show that $r_1''(x_*)<0$.
Similarly, other extreme point is

$$y_*=\left[\frac{q_2(q_2+1)}{(p_2+q_2)(p_2+q_2+1)}\right]^{1/p_2} \tag{32}$$

for $r_2(y)=y^{q_2}\left[q_2+1-(p_2+q_2+1)y^{p_2}\right]$. Then,

$$r_1(x_*)=\left[\frac{q_1(q_1+1)}{(p_1+q_1)(p_1+q_1+1)}\right]^{q_1/p_1}\frac{p_1(q_1+1)}{p_1+q_1}=c_1(p_1,q_1), \tag{33}$$

and

$$r_2(y_*)=\left[\frac{q_2(q_2+1)}{(p_2+q_2)(p_2+q_2+1)}\right]^{q_2/p_2}\frac{p_2(q_2+1)}{p_2+q_2}=c_2(p_2,q_2). \tag{34}$$

1. In Q_1: $\left[\frac{q_1+1}{p_1+q_1+1}\right]^{1/p_1}<x<1$, $\left[\frac{q_2+1}{p_2+q_2+1}\right]^{1/p_2}<y<1$, we have

$$\alpha\ge\frac{-1}{x^{q_1}\left[(p_1+q_1+1)x^{p_1}-(q_1+1)\right]y^{q_2}\left[(p_2+q_2+1)y^{p_2}-(q_2+1)\right]}. \tag{35}$$

Using the critical values, we obtain that in Q_1

$$\alpha\ge\frac{-1}{p_1p_2}. \tag{36}$$

2. In Q_2: $\left[\frac{q_1+1}{p_1+q_1+1}\right]^{1/p_1}<x<1$, $0<y<\left[\frac{q_2+1}{p_2+q_2+1}\right]^{1/p_2}$, we have

$$\alpha\le\frac{1}{x^{q_1}\left[(p_1+q_1+1)x^{p_1}-(q_1+1)\right]y^{q_2}\left[q_2+1-(p_2+q_2+1)y^{p_2}\right]}. \tag{37}$$

Therefore using the analyzing critical values, we have

$$\alpha\le\frac{1}{p_1r_2(y_*)}. \tag{38}$$

3. In Q_3 : $0 < x < \left[\dfrac{q_1+1}{p_1+q_1+1}\right]^{1/p_1}$, $0 < y < \left[\dfrac{q_2+1}{p_2+q_2+1}\right]^{1/p_2}$.

Here, we obtain

$$\alpha \geq \frac{-1}{x^{q_1}\left[q_1+1-(p_1+q_1+1)x^{p_1}\right]y^{q_2}\left[q_2+1-(p_2+q_2+1)y^{p_2}\right]} . \qquad (39)$$

Therefore using the analyzing critical values, we have

$$\alpha \geq \frac{-1}{r_1(x_*)r_2(y_*)} . \qquad (40)$$

4. In Q_4 : $0 < x < \left[\dfrac{q_1+1}{p_1+q_1+1}\right]^{1/p_1}$, $\left[\dfrac{q_2+1}{p_2+q_2+1}\right]^{1/p_2} < y < 1$.

By the analogy with Q_2

$$\alpha \leq \frac{1}{r_1(x_*)p_2} . \qquad (41)$$

Therefore, the admissible range for α which renders equation (22) to be a bivariate d.f. is given by (23). It is evident that equation (22) possesses PQD property for α satisfying (26) with

$$w(x,y) = \alpha x^{q_1+1}\left(1-x^{p_1}\right)y^{q_2+1}\left(1-y^{p_2}\right) \geq 0 . \qquad (42)$$

The theorem thus proved.

The Pearson correlation coefficient of the bivariate joint d.f. given by (22) is

$$\rho = \frac{12\alpha p_1 p_2}{(q_1+2)(p_1+q_1+2)(q_2+2)(p_2+q_2+2)} . \qquad (43)$$

4. FINDINGS AND DISCUSSIONS (BULGULAR VE TARTIŞMALAR)

The data related to yearly producer price index (PPI) and total exports from January 2006 to June 2007 in Turkey are given in Table 1.

Table 1. Yearly producer price index (PPI) and total exports data in Turkey
(Tablo 1. Türkiye'de yıllık üretici fiyatları endeksi (ÜFE) ve toplam ihracat verileri)

Months	Yearly PPI (%)	Total Exports (1000$)
January 2006	5,11	5133049
February 2006	5,26	6058251
March 2006	4,21	7411102
April 2006	4,09	6456090
May 2006	7,66	7041543
June 2006	12,52	7815435
July 2006	14,34	7067411
August 2006	12,32	6811202
September 2006	11,19	7606551
October 2006	10,94	6888813
November 2006	11,67	8641475
December 2006	11,58	8603753
January 2007	9,37	6564560
February 2007	10,13	7656952
March 2007	10,92	8957852
April 2007	9,68	8313312
May 2007	7,14	9147620
June 2007	2,89	8980247

The yearly PPI and total exports are transformed to uniform $[0,1]$. Then the correlation coefficient is $\rho \cong 0.2$.

In equation (22), for $p_1 = p_2 = 2$, $q_1 = q_2 = 1$ and $\alpha \cong 0.9$

$$F(x,y) = xy\{1 + 0.9xy(1 - x^2)(1 - y^2)\} \quad , \quad 0 \le x, y \le 1 . \tag{44}$$

can be used for modeling yearly PPI and total exports.

5. CONCLUSIONS (SONUÇLAR)

In this study, the admissible range of association parameter α is obtained and PQD property is shown. The Pearson correlation coefficient is calculated. In system analysis, reliability theory, survival analysis, finance, etc. the bivariate FGM distributions can be used. It is show that the bivariate FGM distributions can be used as a model in financial data.

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