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THE LAPLACE TRANSFORM OF THE ONE-DIMENSIONAL DISTRIBUTION FUNCTION OF A SEMI-MARKOVIAN RANDOM WALK PROCESS WITH REFLECTING AND DELAYING BARRIERS

ABSTRACT

In this paper, a semi-Markovian random walk process with reflecting barrier on the zero-level and delaying barrier on the $\beta(\beta > 0)$-level is constructed. Furthermore, the Laplace transformation of one-dimensional distribution function of this process is expressed by means of the probability characteristics of random walk $\{Y_n : n \geq 1\}$ and renewal process $\{T_n : n \geq 1\}$.

Keywords: Semi-Markovian Random Walk Process,
           Reflecting Barrier, Delaying Barrier,
           One-Dimensional Distribution Function,
           Laplace Transformation

YANSITAN VE TUTAN BARİYERLİ YARI-MARKOV RASTGELE YÜRÜYÜŞ SÜRECİNİN BİR BOYUTLU DAĞILIM FONKSİYONUNUN LAPLACE DÖNÜŞÜMÜ

ÖZET

Bu çalışmada, sıfır seviyesinde yansıtan ve $\beta(\beta > 0)$ seviyesinde tutan bariyerli bir yarı-Markov rastgele yürüyüş süreci inşa edilmiştir. Ayrıca, bu sürecin bir boyutlu dağılım fonksiyonunun Laplace dönüşümü bir $\{Y_n : n \geq 1\}$ rastgele yürüyüş süreci ile bir $\{T_n : n \geq 1\}$ yenileme sürecinin olasılık karakteristikleri yardımıyla ifade edilmiştir.

Anahtar Kelimeler: Yarı-Markov Rastgele Yürüyüş Süreci,
                  Yansıtan Bariyer, Tutan Bariyer,
                  Bir Boyutlu Dağılım Fonksiyonu,
                  Laplace Dönüşümü

1. INTRODUCTION (GİRİŞ)

It is known that the most of the problems of stock control theory is often given by means of random walks or random walks with delaying barriers(see References 1-5, etc.). But, for the problem considered in this study, one of the barriers is reflecting and the other one is delaying, and the process representing the quantity of the stock has been given by using a random walk and a renewal process. Such models were rarely considered in literature. The practical state of the problem mentioned above is as follows.

Suppose that some quantity of a stock in a certain warehouse is increasing or decreasing in random discrete portions depending to the demands at discrete times. Then, it is possible to characterize the level of stock by a process called the semi-Markovian random walk process. The processes of this type have been widely studied in literature.[1,4 and 6] But sometimes some problems occur in stock control theory such that in order to get an adequate solution we have to consider some processes which are more complex than semi-Markovian random walk processes. For example, if the borrowed quantity is demanded to be added to the warehouse immediately when the qunatiy of demanded stock is more than the total quantity of stock in the warehouse then, it is possible to characterize the level of stock in the warehouse by a stochastic process called as semi-Markovian random walk processes with reflecting barrier. But for the model studied in this study an additional condition has been considered. Since the volume of warehouse is finite in real cases, the supply coming to the warehouse is stopped until the next demand when the warehouse becomes full. In order to characterize the quantity of stock in the warehouse under these conditions it is necessary to use a stochastic process called as semi-Markovian random walk process with two barriers in which one of them is reflecting and the other one is delaying. Note that semi-Markovian random walk processes with two barriers, namely reflecting and delaying, have not been considered enough in literature.[6,7 and 8]

This type problems may ocur, for example, in the control of military stocks, refinery stocks, reserveof oil wells, and etc.

The Model. Assume that we observe random motion of a particle, initially at the position $X_0 \in [0,\beta]$, $\beta > 0$, in a stripe bounded by two barriers; the one lying on the zero-level as reflecting and the other lying on β- level as delaying. Furthermore, assume that this motion proceeds according to the following rules: After staying at the position X_0 for as much as random duration ξ_1, the particle wants to reach the position $X_0 + \eta_1$. If $X_0 + \eta_1 > \beta$ then the particle will be kept at the position $X_1 = \beta$ since there is delaying barrier at β-level. If $X_0 + \eta_1 \in [0,\beta]$, then the particle will be at the position $X_1 = X_0 + \eta_1$. Since there is a reflecting barrier at zero-level, when $X_0 + \eta_1 < 0$ the particle will reflect from this barrier as long as $|X_0 + \eta_1|$. In this case, if $|X_0 + \eta_1| \leq \beta$ then the particle will be kept at the position $X_1 = |X_0 + \eta_1|$ and if $|X_0 + \eta_1| > \beta$ then the particle will be at the position β, so the position of the particle will be $X_1 = min\{\beta, |X_0 + \eta_1|\}$.

After staying at the position X_1 for as much as random duration ξ_2 again it will jump to the position $X_2 = min\{\beta, |X_1 + \eta_2|\}$ according to the above mentioned rules. Thus at the end of n-th jump, the particle will be at the position $X_n = min\{\beta, |X_{n-1} + \eta_n|\}$, $n \geq 1$.

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this study, it is given a semi-Markovian random walk process $X(t)$ with reflecting barrier on the zero-level and delaying barrier on the $\beta(\beta > 0)$-level mathematically. Then the Laplace transformation of one-dimensional distribution function of this process $X(t)$ is expressed by terms of the probability characteristics of a random walk $\{Y_n : n \geq 1\}$ and a renewal process $\{T_n : n \geq 1\}$. Sometimes, it is easier to calculate the Laplace transformation of one-dimensional distribution function of the process $X(t)$ instead of itself.

3. CONSTRUCTION OF THE PROCESS (SÜRECİN KURULMASI)

Suppose $\{(\xi_i, \eta_i)\}$, $i = 1, 2, 3, \ldots$ is a sequence of identically and independently distributed pairs of random variables, defined in any probability space $(\Omega, \mathcal{F}, P)$ such that ξ_i's are positive valued, i.e., $P\{\xi_i > 0\} = 1$, $i = 1, 2, 3, \ldots$. Also let us denote the distribution function of ξ_i and η_i

$$\Phi(t) = P\{\xi_1 < t\}, \quad F(x) = P\{\eta_1 < x\}, \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R},$$

respectively. Before stating the corresponding process let us construct the following sequences of random variables:

$$T_n = \sum_{i=1}^n \xi_i, \quad Y_n = \sum_{i=1}^n \eta_i, \quad n \geq 1, T_0 = Y_0 = 0.$$

Then the processes $\{T_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$ forms a renewal process and a random walk respectively. By using the random pairs (ξ_i, η_i) we can construct the random walk process with two barriers in which the reflecting barrier is on the zero-level and the delaying barrier is on $\beta > 0$-level as follows:

$$X_n = min\{\beta, |X_{n-1} + \eta_n|\}, \quad n \geq 1, \quad z = X_0 \in [0, \beta],$$

where z is the initial position of the process. Now, let us construct the stochastic process $X(t)$ which has the reflecting barrier from below and the delaying barrier from above and which represents the level of stock at the moment t:

$$X(t) = X_n, \quad \text{if} \quad t \in [T_n, T_{n+1}).$$

This process is called the semi-Markovian random walk with the reflecting barrier on the zero-level and the delaying barrier on β-level.

4. ONE-DIMENSIONAL DISTRIBUTION FUNCTION OF THE PROCESS $X(t)$
($X(t)$ SÜRECİNİN BİR BOYUTLU DAĞILIM FONKSİYONU)

In order to formulate the main results of this paper, let us state the following probability characteristics of random walk $\{Y_n : n \geq 1\}$ and renewal process $\{T_n : n \geq 1\}$:

$$a_n(z, x) = P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n; z + Y_n \in [0, x]\}, \quad n \geq 1,$$

$$b_n(z, v) = P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n - 1; z + Y_n < v\}, \quad v \geq 0, \quad n \geq 1,$$

$$A(t, z, x) = \sum_{n=0}^\infty a_n(z, x) \Delta \Phi_n(t),$$

$$C(s, z, v) = \sum_{n=0}^\infty c_n(z, v) \Delta \Phi_n(s),$$

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$$C(ds, z, dv) = \sum_{n=0}^{\infty} c_n(z, dv) \, d\Phi_n(s), \quad c_n(z, dv) = d_v c_n(z, v),$$

$$\Phi_n(t) = P\{T_n < t\},$$

$$\Delta\Phi_n(t) = \Phi_n(t) - \Phi_{n+1}(t),$$

where $z = X_0 \in [0, \beta]$, $v < 0$, $n \geq 1$ and $a_n(z, x) = 1$, $b_n(z, v) = 0$.

For any function $M(t, z)$, let us denote the Laplace transformation and Laplace-Stieltjes transformation of $M(t, z)$

$$\tilde{M}(\lambda, z) = \int_0^{\infty} e^{-\lambda t} M(t, z) dt$$

and

$$\tilde{M}(\lambda, z) = \int_0^{\infty} e^{-\lambda t} d_t M(t, z)$$

with respect to t, respectively. Moreover, for any functions $M_1(t, z)$ and $M_2(t, z)$, the convolution product of $M_1(t, z)$ and $M_2(t, z)$ is given as follows:

$$M_1(t, z) * M_2(t, z) = \int_0^t M_2(t - s, z) \, d_s M_1(s, z)$$

and m- times convolution product of $M_1(t, z)$ with itself.

$$[M_1(t, z)]_*^k = M_1(t, z) * [M_1(t, z)]_*^{k-1}.$$

Let $Q(t, z, x)$ denote the one-dimensional distribution function of the process $X(t)$, that is

$$Q(t, z, x) = P_z\{X(t) < x\} = P\{X(t) < x : X(0) = z\}; \quad x, z \in [0, \beta], t \in \mathbb{R}^+.$$

Firstly, let us give the following lemma. For this aim, we denote by γ the first falling moment of the process $X(t)$ into the delaying barrier. As associated with this we define $R(t, z, x)$ as the following conditional distribution:

$$R(t, z, x) = P_z\{\gamma \geq t : X(t) < x\}.$$

Lemma 4.1: If ξ_1 and η_1 are independent random variables in the initial sequence of random pairs mentioned above, then the Laplace transformation of the conditional distribution $R(t, z, x)$ is given in terms of the probability characteristics of random walk $\{Y_n : n \geq 1\}$ and renewal process $\{T_n : n \geq 1\}$, as follows:

$$\tilde{R}(\lambda, z, x) = \tilde{A}(\lambda, z, x) + \sum_{m=1}^{\infty} \int_{-\beta}^0 \cdots \int_{-\beta}^0 \prod_{i=1}^m C^*(\lambda, |v_{i-1}|, x) \tilde{A}(\lambda, |v_m|, x)$$

where $|v_0| = z \in [0, \beta]$.

Proof: Let us denote by $v_0(t)$ the number of reflections of the process $X(t)$ into the interval $[0, t]$. According to the total probability formula we have

$$R(t, z, x) = P_z\{\gamma \geq t : X(t) < x\} = \sum_{m=0}^{\infty} P_z\{\gamma \geq t : v_0(t) = m; X(t) < x\}$$

and to make it simple, we put

$$r_m(t, z, x) = P_z\{\gamma \geq t : v_0(t) = m; X(t) < x\}, m \geq 0.$$

Now, we expres each $r_m(t, z, x)$ by the probability characteristics of both $\{Y_n : n \geq 1\}$ and $\{T_n : n \geq 1\}$, seperately. Therefore, we can write

$$r_0(t, z, x) = P_z\{\gamma \geq t : v_0(t) = 0; X(t) < x\}$$

$$= \sum_{n=0}^{\infty} P_z\{\gamma \geq t: v_0(t) = 0; X(t) < x; T_n \leq t < T_{n+1}\}$$

$$= \sum_{n=0}^{\infty} P\{z + Y_i \in [0,\beta]; 1 \leq i \leq n; z + Y_n \in [0,x]\} P\{T_n \leq t < T_{n+1}\}$$

$$= \sum_{n=0}^{\infty} a_n(z,x)\Delta\Phi_n(t)$$

$$= A(t,z,x)$$

Thus, we get $\widetilde{r_0}(\lambda,z,x) = \tilde{A}(\lambda,z,x)$. Now, we can calculate the conditional distribution $r_1(t,z,x)$:

$$r_1(t,z,x) = P_z\{\gamma \geq t: v_0(t) = 1; X(t) < x\}$$

$$= \sum_{n=0}^{\infty} P_z\{\gamma \geq t: v_0(t) = 1; X(t) < x; T_n \leq t < T_{n+1}\}$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n} \int_0^t P_z\{\gamma \geq t: v_0(t) = 1; X(t) < x; T_n \leq t < T_{n+1}; T_i \in ds\}$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n} \int_{-\beta}^0 \int_0^t P\{z + Y_j \in [0,\beta]; 1 \leq j \leq i-1; z + Y_i \in dv; T_i \in ds\}$$
$$\times P\{|v| + Y_j \in [0,\beta]; 1 \leq j \leq n-i; |v| + Y_{n-i} < x\}\Delta\Phi_{n-i}(t-s)$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n} \int_{-\beta}^0 \int_0^t c_i(z,dv)d\Phi_i(s)a_{n-i}(|v|,x)\Delta\Phi_{n-i}(t-s)$$

$$= \int_{-\beta}^0 \int_0^t \sum_{i=1}^{\infty} c_i(z,dv)d\Phi_i(s) \sum_{n=i}^{\infty} a_{n-i}(|v|,x)\Delta\Phi_{n-i}(t-s)$$

$$= \int_{-\beta}^0 \int_0^t C(ds,z,dv) A(t-s,|v|,x)$$

$$= \int_{-\beta}^0 A(t,|v|,x) * C(t,z,dv)$$

As such,

$$\widetilde{r_1}(t,z,x) = \int_{-\beta}^0 \tilde{A}(\lambda,|v|,x)C^*(\lambda,z,dv)$$

is obtained. Analogously, it is possible to prove that

$$\widetilde{r_m}(t,z,x) = \int_{-\beta}^0 \cdots \int_{-\beta}^0 \prod_{i=1}^{m} C^*(\lambda,|v_{i-1}|,dv_i)\tilde{A}(\lambda,|v_m|,x), m \geq 1; |v_0| = z \in [0,\beta].$$

Substituting all of these expressions in the formula for $\tilde{R}(\lambda,z,x)$ given above, we have

$$\tilde{R}(\lambda,z,x) = \tilde{A}(\lambda,z,x) + \sum_{m=1}^{\infty} \int_{-\beta}^0 \cdots \int_{-\beta}^0 \prod_{i=1}^{m} C^*(\lambda,|v_{i-1}|,x)\tilde{A}(\lambda,|v_m|,x)$$

as asserted. Hence the lemma is proved.

Now, we can formulate the main result of this section as the following theorem.

Theorem 4.1: Under the conditions of Lemma 4.1, in terms of the probability characteristics of renewal process $\{T_n : n \geq 1\}$, and random walk $\{Y_n : n \geq 1\}$, the Laplace transformation of one-dimensional distribution function of the process $X(t)$, $\tilde{Q}(\lambda, z, x)$, can be given as follows:

$$\tilde{Q}(\lambda, z, x) = \frac{\tilde{R}(\lambda, \beta, x) + \lambda.\left[\tilde{R}(\lambda, z, x)\tilde{R}(\lambda, \beta, \infty) - \tilde{R}(\lambda, \beta, x)\tilde{R}(\lambda, \beta, \infty)\right]}{\lambda.\tilde{R}(\lambda, \beta, \infty)},$$

where $\tilde{R}(\lambda, \beta, \infty) = \lim_{x \to \infty} \tilde{R}(\lambda, \beta, x)$.

Proof: According to the total probability Formula, we get

$$Q(t, z, x) = P_z\{X(t) < x\} = P_z\{\gamma \geq t : X(t) < x\} + P_z\{\gamma < t : X(t) < x\}.$$

On the other hands, we can write

$$P_z\{\gamma < t : X(t) < x\} = \int_0^t P_z\{\gamma \in ds; X(t) < x\}$$
$$= \int_0^t P_z\{\gamma \in ds\}\, Q(t - s, \beta, x).$$

Therefore it is given

$$Q(t, z, x) = R(t, z, x) + \int_0^t Q(t - s, \beta, x)\, H(ds, z),$$

where $H(t, z) = P_z\{\gamma < t\}$, so that, $H(ds, z) = P_z\{\gamma \in ds\}$. Thus we have

$$Q(t, z, x) = R(t, z, x) + H(t, z) * Q(t, \beta, x),$$

that is,

$$\tilde{Q}(\lambda, z, x) = \tilde{R}(\lambda, z, x) + H^*(\lambda, z)\tilde{Q}(\lambda, \beta, x).$$

If we subsitute β for z, then we get

$$\tilde{Q}(\lambda, \beta, x) = \tilde{R}(\lambda, z, x) + H^*(\lambda, \beta)\tilde{Q}(\lambda, \beta, x),$$

or with an other words,

$$\tilde{Q}(\lambda, \beta, x)[1 - H^*(\lambda, \beta)] = \tilde{R}(\lambda, z, x).$$

Thus, we can write

$$\tilde{Q}(\lambda, \beta, x) = \tilde{R}(\lambda, z, x)[1 - H^*(\lambda, \beta)]^{-1}.$$

By substituting this exression in the formula for $\tilde{Q}(\lambda, z, x)$ given above, we obtain

$$\tilde{Q}(\lambda, z, x) = \tilde{R}(\lambda, z, x) + H^*(\lambda, z)\tilde{R}(\lambda, \beta, x)[1 - H^*(\lambda, \beta)]^{-1}$$
$$= \frac{\tilde{R}(\lambda, z, x) + H^*(\lambda, z)\tilde{R}(\lambda, \beta, x) - H^*(\lambda, z)\tilde{R}(\lambda, z, x)}{1 - H^*(\lambda, \beta)}.$$

On the other hands, since $H(t, z) = 1 - R(t, z, \infty)$, it is easy to see that

$$H^*(\lambda, z) = 1 - R^*(\lambda, z, \infty) = 1 - \lambda\tilde{R}(\lambda, z, \infty),$$

or equivalently,

$$H^*(\lambda, \beta) = 1 - \lambda\tilde{R}(\lambda, \beta, \infty).$$

Thus we have

$$\tilde{Q}(\lambda, z, x) = \frac{\tilde{R}(\lambda, \beta, x) + \lambda.\left[\tilde{R}(\lambda, z, x)\tilde{R}(\lambda, \beta, \infty) - \tilde{R}(\lambda, \beta, x)\tilde{R}(\lambda, \beta, \infty)\right]}{\lambda.\tilde{R}(\lambda, \beta, \infty)},$$

as asserted. Hence the theorem is proved.

5. CONCLUSIONS AND RECOMMENDATIONS (SONUÇ VE ÖNERİLER)

In this paper, the semi-Markovian random walk process $X(t)$, with reflecting barrier on the zero-level and delaying barrier on the $\beta(\beta > 0)$-level, is given. The Laplace transformation of one-dimensional distribution function of this process is expressed by means of the probability characteristics of a random walk $\{Y_n : n \geq 1\}$ and a renewal

process $\{T_n : n \geq 1\}$. If the Laplace transformation of one-dimensional distribution function of process $X(t)$ is known, then it is possible to give the explicit expressions for the expected value and the higher order moments of it by means of this Laplace transformation.

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