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**THE LIMIT BEHAVIOUR OF SEMI-MARKOVIAN RANDOM WALK PROCESSES WITH
REFLECTING AND DELAYING BARRIERS**

ABSTRACT

In this paper, a semi-Markovian random walk process with reflecting barrier on the zero-level and delaying barrier on the $\beta(\beta > 0)$ -level is constructed. Furthermore, the limit theorems for the sequence of this type processes have been proved and one example has been given.

Keywords: Semi-Markovian Random Walk Process,
Reflecting Barrier, Delaying Barrier,
Limit Theorem, Wiener process.

**YANSITAN VE TUTAN BARIYERLİ YARI-MARKOV RASTGELE YÜRÜYÜŞ SÜREÇLERİNİN
LİMİT DAVRANIŞI**

ÖZET

Bu çalışmada, sıfır seviyesinde yansıtıcı ve $\beta(\beta > 0)$ -seviyesinde tutan bariyerli bir yarı-Markov rastgele yürüyüş süreci inşa edilmiştir. Ayrıca, bu tipten süreçler dizisi için limit teoremleri ispatlanmış ve bir örnek verilmiştir.

Anahtar Kelimeler: Yarı-Markov Rastgele Yürüyüş Süreci,
Yansıtıcı Bariyer, Tutan Bariyer,
Limit Teoremi, Wiener Süreci.



1. INTRODUCTION (GİRİŞ)

It is known that the most of the problems of stock control theory is often given by means of random walks or random walks with delaying barriers(see References [1, 2, 3, 4 and 5],etc.). But, for the problem considered in this study, one of the barriers is reflecting and the other one is delaying, and the process representing the quantity of the stock has been given by using a random walk and a renewal process. Such models were rarely considered in literature. The practical state of the problem mentioned above is as follows.

Suppose that some quantity of a stock in a certain warehouse is increasing or decreasing in random discrete portions depending to the demands at discrete times. Then, it is possible to characterize the level of stock by a process called the semi-Markovian random walk process. The processes of this type have been widely studied in literature (see, References [1, 4 and 6],etc.). But sometimes some problems occur in stock control theory such that in order to get an adequate solution we have to consider some processes which are more complex than semi-Markovian random walk processes. For example, if the borrowed quantity is demanded to be added to the warehouse immediately when the quantity of demanded stock is more than the total quantity of stock in the warehouse then, it is possible to characterize the level of stock in the warehouse by a stochastic process called as semi-Markovian random walk processes with reflecting barrier. But for the model studied in this study an additional condition has been considered. Since the volume of warehouse is finite in real cases, the supply coming to the warehouse is stopped until the next demand when the warehouse becomes full. In order to characterize the quantity of stock in the warehouse under these conditions it is necessary to use a stochastic process called as semi-Markovian random walk process with two barriers in which one of them is reflecting and the other one is delaying. Note that semi-Markovian random walk processes with two barriers, namely reflecting and delaying, have not been considered enough in literature (see, for example References [6, 7, 8 and 9]).

This type problems may occur, for example, in the control of military stocks, refinery stocks, reserve of oil wells, and etc.

The Model. Assume that we observe random motion of a particle, initially at the position $X_0 \in [0, \beta]$, $\beta > 0$, in a stripe bounded by two barriers; the one lying on the zero-level as reflecting and the other lying on β -level as delaying. Furthermore, assume that this motion proceeds according to the following rules: After staying at the position X_0 for as much as random duration ξ_1 , the particle wants to reach the position $X_0 + \eta_1$. If $X_0 + \eta_1 > \beta$ then the particle will be kept at the position $X_1 = \beta$ since there is a delaying barrier at β -level. If $X_0 + \eta_1 \in [0, \beta]$, then the particle will be at the position $X_1 = X_0 + \eta_1$. Since there is a reflecting barrier at zero-level, when $X_0 + \eta_1 < 0$ the particle will reflect from this barrier as long as $|X_0 + \eta_1|$. In this case, if $|X_0 + \eta_1| \leq \beta$ then the particle will be kept at the position $X_1 = |X_0 + \eta_1|$ and if $|X_0 + \eta_1| > \beta$ then the particle will be at the position β , so that the particle will be kept at the position $X_1 = \min\{\beta, |X_0 + \eta_1|\}$.

After staying at the position X_1 for as much as random duration ξ_2 again it will jump to the position $X_2 = \min\{\beta, |X_1 + \eta_2|\}$ according to the above mentioned rules. Thus at the end of n-th jump, the particle will be at the position $X_n = \min\{\beta, |X_{n-1} + \eta_n|\}$, $n \geq 1$.

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this study, it is given a semi-Markovian random walk process with reflecting barrier on the zero-level and delaying barrier on the $\beta(\beta > 0)$ – level mathematically. Then the limit theorems for the sequence of this type processes have been proved and one example has been given.

3. CONSTRUCTION OF THE PROCESS (SÜRECİN KURULMASI)

Suppose $\{(\xi_i, \eta_i)\}$, $i = 1, 2, 3, \dots$ is a sequence of identically and independently distributed pairs of random variables, defined in any probability space (Ω, \mathcal{F}, P) such that ξ_i 's are positive valued, i.e., $P\{\xi_i > 0\} = 1$, $i = 1, 2, 3, \dots$. Also let us denote the distribution function of ξ_i and η_i

$$\Phi(t) = P\{\xi_1 < t\}, \quad F(x) = P\{\eta_1 < x\}, \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}$$

respectively. Before stating the corresponding process let us construct the following sequences of random variables:

$$T_k = \sum_{i=1}^k \xi_i, \quad Y_k = \sum_{i=1}^k \eta_i, \quad k \geq 1; \quad T_0 = Y_0 = 0$$

and

$$v(t) = \inf\{k \geq 0: T_{k+1} > t\}, \quad t > 0; \quad v(0) = 0.$$

Note that the processes $\{T_n: n \geq 1\}$ and $\{Y_n: n \geq 1\}$ forms a renewal process and a random walk respectively. By using the random pairs (ξ_i, η_i) we can construct the semi-Markovian random walk process $\chi(t)$ as follows:

$$\chi(t) = \sum_{i=1}^k \eta_i, \quad \text{if } T_k \leq t < T_{k+1}$$

or

$$\chi(t) = \sum_{i=1}^{v(t)} \eta_i. \tag{3.1}$$

This process forms a semi-Markovian random walk without barriers, where $v(t)$ denotes the numbers of jumps the process $\chi(t)$ during the time interval $[0, t]$. we restrict the process $\chi(t)$ at the level zero by a reflecting barrier. Let us denote

$$v_1 = \inf\{m \geq 1: \chi(T_m) \leq 0\}.$$

Then v_1 denotes the number of the steps of the process $\chi(t)$ goes down the zero level. Let us denote the first falling moment of the process $\chi(t)$ on the zero level by T_{v_1} . Thus

$$\gamma_1 = T_{v_1} \equiv \xi_1 + \xi_2 + \dots + \xi_{v_1}$$

and we construct random variable ψ_1 as

$$\psi_1 = |\chi(T_{v_1})| = |\chi(\gamma_1)|.$$

Now, let's define random variable v_2 by

$$v_2 = \inf\{m \geq v_1: \psi_1 + \eta_{v(\gamma_1)+1} + \dots + \eta_{v(T_m)} \leq 0\}.$$

Then let us denote

$$\gamma_2 = T_{v_2} \equiv \xi_1 + \xi_2 + \dots + \xi_{v_1} + \xi_{v_1+1} + \dots + \xi_{v_2}$$

and we construct random variable ψ_2

$$\psi_2 = |\psi_1 + \eta_{v(\gamma_1)+1} + \dots + \eta_{v(\gamma_2)}|.$$

Analogously, let us denote

$$v_k = \inf\{m \geq v_{k-1}: \psi_{k-1} + \eta_{v(\gamma_{k-1})+1} + \dots + \eta_{v(T_m)} \leq 0\}, \quad k \geq 1.$$

Also

$$\gamma_k = T_{v_k} \equiv \xi_1 + \xi_2 + \dots + \xi_{v_k}, \quad k \geq 1, \quad \gamma_0 = 0.$$

Thus we construct a random variable ψ_k

$$\psi_k = |\psi_{k-1} + \eta_{v(\gamma_{k-1})+1} + \dots + \eta_{v(\gamma_k)}| = |\psi_{k-1} + \chi(\gamma_k) - \chi(\gamma_{k-1})|$$

where $k \geq 1, \psi_0 = 0$. Now, we construct random variable $\mu(t)$ with integer valued

$$\mu(t) = \inf\{k \geq 0: \gamma_{k+1} > t\}, \quad t > 0; \quad \mu(0) = 0.$$



Then $\mu(t)$ can be written as

$$\mu(t) = k, \text{ if } \gamma_k \leq t < \gamma_{k+1}.$$

By using random variable $\mu(t)$, we define following processes

$$\gamma(t) = \gamma_{\mu(t)},$$

$$\psi(t) = \psi_k, \text{ if } \gamma_k \leq t < \gamma_{k+1}$$

or

$$\begin{aligned} \psi(t) &= \psi_{\mu(t)} = \left| \psi_{\mu(t)-1} + \eta_{\nu(\gamma_{\mu(t)-1}+1)} + \dots + \eta_{\nu(\mu(t))} \right| \\ &= \left| \psi_{\mu(t)-1} + \chi(\gamma_{\mu(t)}) - \chi(\gamma_{\mu(t)-1}) \right| \end{aligned}$$

and

$$\tilde{\chi}(t) = \psi_{\mu(t)} + \eta_{\nu(\gamma_{\mu(t)})+1} + \dots + \eta_{\nu(t)}.$$

Then, we can write $\tilde{\chi}(t)$ as follows:

$$\tilde{\chi}(t) = \psi_{\mu(t)} + \chi(t) - \chi(\gamma_{\mu(t)}) = \psi(t) + \chi(t) - \chi(\gamma(t)). \quad (3.2)$$

The process $\tilde{\chi}(t)$ forms a semi-Markovian random walk process with reflecting barrier on the zero level.

Now, we delay the process $\tilde{\chi}(t)$ at $\beta(\beta > 0)$ -level. The general form the processes of semi-Markovian random walk with delaying barrier on the zero level is given by Feller (1968). If $\alpha(t)$ is any process, then the process $\alpha_0^*(t)$ with delaying barrier on the zero level is given as

$$\alpha_0^*(t) = \alpha(t) - \inf_{0 \leq s \leq t} \{0, \alpha(s)\}.$$

In a similar way, we obtain the process with delaying barrier on the $\beta(\beta > 0)$ -level:

$$\alpha_\beta^*(t) = \beta + \alpha(t) - \sup_{0 \leq s \leq t} \{\beta, \alpha(s)\}.$$

Thus, we can write

$$X(t) = \alpha_\beta^*(t) = \beta + \tilde{\chi}(t) - \sup_{0 \leq s \leq t} \{\beta, \tilde{\chi}(s)\}. \quad (3.3)$$

The process $X(t)$ form a semi-Markovian random walk with delaying barrier on the $\beta(\beta > 0)$ -level and reflecting barrier on the zero level. By using (3.2) and (3.3), we can write

$$X(t) = \beta + \psi(t) + \chi(t) - \chi(\gamma(t)) - \sup_{0 \leq s \leq t} \{\beta, \psi(s) + \chi(s) - \chi(\gamma(s))\}.$$

On the other hand, we construct the sequence of the processes of semi-Markovian random walk with reflecting barrier on the zero level and delaying barrier on the $\beta(\beta > 0)$ -level. Let is given the sequence of series in which every series is independently and identically distributed pairs of random variables $\{(\xi_i^{(n)}, \eta_i^{(n)})\}$, $i = 1, 2, 3, \dots$; $n = 1, 2, 3, \dots$, where random variables $\xi_i^{(n)}$ are positive. Suppose that the random variables $\xi_i^{(n)}$ and $\eta_i^{(n)}$ are independent such that

$$\lim_{n \rightarrow \infty} (P\{\xi_1^{(n)} > \varepsilon\} + P\{\eta_1^{(n)} > \varepsilon\}) = 0$$

where $\varepsilon > 0$.

By using $\{(\xi_i^{(n)}, \eta_i^{(n)})\}$, $i = 1, 2, 3, \dots$; $n = 1, 2, 3, \dots$, we construct the sequence of the semi-Markovian random walk processes $\chi_n(t)$ as follows:

$$\chi_n(t) = \sum_{i=1}^{\nu_n(t)} \eta_i^{(n)}, t > 0; \chi_n(0) = 0$$

where

$$\nu_n(t) = \inf\{k \geq 0: T_{k+1}^{(n)} > t\}$$

or

$$\nu_n(t) = k, \text{ if } T_k^{(n)} \leq t < T_{k+1}^{(n)}; T_k^{(n)} = \sum_{i=1}^k \xi_i^{(n)}.$$

We define the sequence of random variables $\nu_1^{(n)}, \nu_2^{(n)}, \dots, \nu_k^{(n)}, \dots$ as follows:
 Let

$$\nu_1^{(n)} = T_{\nu_1^{(n)}}^{(n)} = \xi_1^{(n)} + \xi_2^{(n)} + \dots + \xi_{\nu_1^{(n)}}^{(n)}$$

and

$$\psi_1^{(n)} = \left| \chi_n \left(T_{v_1^{(n)}}^{(n)} \right) \right| = \left| \chi_n(\gamma_1^{(n)}) \right|$$

where

$$v_1^{(n)} = \inf \{ m \geq 1 : \chi_n(T_m) \leq 0 \}.$$

Analogously, we define

$$\gamma_k^{(n)} = T_{v_k^{(n)}}^{(n)} = \xi_1^{(n)} + \xi_2^{(n)} + \dots + \xi_{v_k^{(n)}}^{(n)}, k \geq 1; \gamma_0^{(n)} = 0$$

and

$$\psi_k^{(n)} = \left| \psi_{v_{k-1}^{(n)}}^{(n)} + \eta_{v_n(\gamma_{k-1}^{(n)})+1}^{(n)} + \dots + \eta_{v_n(\gamma_k^{(n)})}^{(n)} \right|$$

where

$$v_k^{(n)} = \inf \left\{ m \geq v_{k-1}^{(n)} : \psi_{v_{k-1}^{(n)}}^{(n)} + \eta_{v_n(\gamma_{k-1}^{(n)})+1}^{(n)} + \dots + \eta_{v_n(T_m^{(n)})}^{(n)} \leq 0 \right\}, k \geq 1.$$

Also, we construct the following processes:

$$\begin{aligned} \mu_n(t) &= k, \text{ if } \gamma_k^{(n)} \leq t < \gamma_{k+1}^{(n)}, \\ \gamma_n(t) &= \gamma_k^{(n)}, \\ \psi_n(t) &= \psi_k^{(n)}, \text{ if } \gamma_k^{(n)} \leq t < \gamma_{k+1}^{(n)}. \end{aligned}$$

Then we have

$$\begin{aligned} \psi_n(t) = \psi_{\mu_n(t)}^{(n)} &= \left| \psi_{\mu_n(t)-1}^{(n)} + \eta_{v_n(\gamma_{\mu_n(t)-1}^{(n)})+1}^{(n)} + \dots + \eta_{v_n(\gamma_{\mu_n(t)}^{(n)})}^{(n)} \right| \\ &= \left| \psi_{\mu_n(t)-1}^{(n)} + \chi_n \left(\gamma_{\mu_n(t)}^{(n)} \right) - \chi_n \left(\gamma_{\mu_n(t)-1}^{(n)} \right) \right|. \end{aligned}$$

If we define

$$\tilde{\chi}_n(t) = \psi_n(t) + \eta_{v_n(\gamma_n(t))+1}^{(n)} + \dots + \eta_{v_n(\gamma_n(t))}^{(n)},$$

then we can write $\tilde{\chi}_n(t)$ as follows:

$$\tilde{\chi}_n(t) = \psi_{\mu_n(t)}^{(n)} + \chi_n(t) - \chi_n \left(\gamma_{\mu_n(t)}^{(n)} \right) = \psi_n(t) + \chi_n(t) - \chi_n(\gamma_n(t)) \quad (3.4)$$

Thus, we constructed the sequence of processes of semi-Markovian random walk with reflecting barrier on the zero-level. If we delay the process $\tilde{\chi}_n(t)$ at $\beta(\beta > 0)$ -level, then we get

$$\begin{aligned} X_n(t) = \tilde{\chi}_n^* &= \beta + \tilde{\chi}_n(t) - \sup_{0 \leq s \leq t} \{ \beta, \tilde{\chi}_n(s) \} \\ &= \beta + \psi_n(t) + \chi_n(t) - \chi_n(\gamma_n(t)) \\ &\quad - \sup_{0 \leq s \leq t} \{ \beta, \psi_n(s) + \chi_n(s) - \chi_n(\gamma_n(s)) \}. \end{aligned}$$

The processes $X_n(t)$ form semi-Markovian random walks with reflecting barrier on the zero-level and the delaying barrier is on $\beta(\beta > 0)$ -level. Our aim is to find the limit process of $X_n(t)$ as $n \rightarrow \infty$.

4. THE LIMIT THEOREM FOR THE PROCESSES (SÜREÇLER İÇİN LİMİT TEOREMİ)

Assume that $\gamma(t)$ and $\psi(t)$ are as given before. Then we can rewrite $\gamma(t)$ as follows:

$$\gamma(t) = \sum_{k=1}^{\mu(t)} \tilde{\gamma}_k, \quad (4.1)$$

where $\tilde{\gamma}_k = \gamma_{k+1} - \gamma_k, k \geq 0$, are independently and identically distributed random variables with positive valued. Thus, we have

$$\gamma(t) = \sum_{k:k \leq t} \tilde{\gamma}_k. \quad (4.2)$$

This process is a positive process with independent increments. If we consider the construction of the process $\tilde{\chi}(t)$, then we see that

$$\psi(t) = \psi_{\mu(t)} = \tilde{\eta}_{v(\mu(t))}, \quad (4.3)$$

where $\tilde{\eta}_{v(\mu(t))}$ is the reflecting part of random variable $\eta_{v(\mu(t))}$ from the reflecting barrier. It was proved that the sequence of the processes $\chi_n(t)$ converges some process $Y(t)$ under the certain conditions by Nasirova (1984). Therefore, firstly, we must prove the theorem about



convergence of the sequence of the processes $\tilde{\chi}_n(t)$. From (3.4) we know that

$$\tilde{\chi}_n(t) = \psi_n(t) + \chi_n(t) - \chi_n(\gamma_n(t)).$$

According to (4.3) we can write

$$\psi_n(t) = \tilde{\eta}_{\nu_n(\gamma_n(t))}^{(t)}$$

where $\tilde{\eta}_{\nu_n(\gamma_n(t))}^{(t)}$ is reflecting part of random variable $\eta_{\nu_n(\gamma_n(t))}^{(t)}$. Then we can rewrite $\tilde{\chi}_n(t)$ in the following form:

$$\tilde{\chi}_n(t) = \tilde{\eta}_{\nu_n(\gamma_n(t))}^{(t)} + \chi_n(t) - \chi_n(\gamma_n(t)) \quad (4.4)$$

Theorem 4.1: If the sequence of the processes $\chi_n(t)$ is bounded on probability and finitely divisible distributions of $\chi_n(t)$ converges to finitely divisible distribution of some process $Y(t)$, then the limit process $\tilde{\chi}(t)$ of the sequence of the processes $\tilde{\chi}_n(t)$ has the following form:

$$\tilde{\chi}(t) = Y(t) - Y(\gamma'(t))$$

where $Y(t) = \eta(\xi^{-1}(t))$, $\xi(t)$ and $\gamma'(t)$ are positive precipitously and creasingly process with independent increments, $\eta(t)$ is homogeneous process with independent increments and

$$\xi^{-1}(t) = s \text{ if } \xi(s-0) \leq t < \xi(s).$$

Proof: According to (4.2) we write

$$\gamma(t) = \sum_{k:k \leq N_n t} \tilde{\gamma}_k^{(n)},$$

where $\tilde{\gamma}_k^{(n)} = \gamma_k^{(n)} - \gamma_{k-1}^{(n)}$ are independently and identically distributed positive random variables, $N_n \rightarrow \infty$ when $n \rightarrow \infty$. Then we can shown that there is a sequence of numbers n_m such that $\gamma_{n_m}(t)$ converges on distribution to some process $\gamma'(t)$, (see Gihman and Skorohod (1975)). The process $\gamma'(t)$ will be positive precipitously and creasingly process with independent increments. It is obvious that

$$\chi_n(t) = \sum_{i=1}^{\nu_n(t)} \eta_i^{(n)}, \quad \xi_n(t) = \sum_{i:i \leq N_n t} \xi_i^{(n)} \quad \text{and} \quad \eta_n(t) = \sum_{i:i \leq N_n t} \eta_i^{(n)}.$$

Then we can write

$$\chi_n(t) = \eta_n(\xi_n^{-1}(t))$$

where $\xi_n^{-1}(t) = s$ if $\xi_n(s-0) \leq t < \xi_n(s)$. Therefore (3.4) can be rewritten as

$$\tilde{\chi}_n(t) = \tilde{\eta}_{\nu_n(\gamma_n(t))}^{(t)} + \eta_n(\xi_n^{-1}(t)) - \eta_n(\xi_n^{-1}(\gamma_n(t))),$$

where $\tilde{\eta}_{\nu_n(\gamma_n(t))}^{(t)}$ is the reflecting part of random variable $\eta_{\nu_n(\gamma_n(t))}^{(t)}$. Thus $\tilde{\eta}_{\nu_n(\gamma_n(t))}^{(t)} \rightarrow 0$ on probability when $n \rightarrow \infty$. By the construction, the processes $\gamma_n(t)$ and $\xi_n(t)$ are identically. Then the limit process $\xi(t)$ will be also positive precipitously and creasingly process with independent increments. The process $\eta_n(t)$ is sum of independently and identically distributed random variables $\eta_k^{(n)}$ which are small infinitely. So that $\eta_n(t)$ will be converges to homogeneous process $\eta(t)$ with independent increments.

According to the condition of the theorem and the equation (3.4) we have

$$\eta_n(\xi_n^{-1}(t)) \rightarrow \eta(\xi^{-1}(t))$$

and

$$\eta_n(\xi_n^{-1}(\gamma_n(t))) \rightarrow \eta(\xi^{-1}(\gamma'(t)))$$

on distribution when $n \rightarrow \infty$. So, the limit process $\tilde{\chi}(t)$ of the sequence $\tilde{\chi}_n(t)$ is as follows:

$$\tilde{\chi}(t) = Y(t) - Y(\gamma'(t)).$$

Thus the theorem is proved. Now, we state the following theorem.

Theorem 4.2: If the sequence of processes $\chi_n(t)$ is bounded on probability and finitely divisible distributions of $\chi_n(t)$ converges to finitely divisible distributions of some process $Y(t)$, then the limit process $\tilde{\chi}^*(t)$ of the sequence of the processes $\tilde{\chi}_n^*(t)$ has the following form:

$$\tilde{\chi}^*(t) = \beta + Y(t) - Y(\gamma'(t)) - \sup_{0 \leq s \leq t} \{\beta, Y(s) - Y(\gamma'(s))\}$$

where $Y(t)$ and $\gamma'(t)$ is as given in Theorem 4.1.

Proof: The transformation

$$\varphi(y(t)) = \beta + y(t) - \sup_{0 \leq s \leq t} \{\beta, y(s)\}$$

is continuous in the J-topology (see Gihman and Skorohod (1975)) and transforms the space D into D , where D is space of functions which haven't any discontinuity of the second type. Therefore, for some functional $F(y(\cdot))$ which is J-continuous on D , $F_1(y(\cdot)) = F(\varphi(y(\cdot)))$ will be also J-continuous. Consequently the distribution $F_1(\tilde{\chi}_n^*(\cdot))$ converges the distribution $F_1(\tilde{\chi}^*(\cdot))$. Thus, we have

$$\begin{aligned} \tilde{\chi}^*(t) &= \beta + \tilde{\chi}(t) - \sup_{0 \leq s \leq t} \{\beta, \tilde{\chi}(s)\} \\ &= \beta + Y(t) - Y(\gamma'(t)) - \sup_{0 \leq s \leq t} \{\beta, Y(s) - Y(\gamma'(s))\} \end{aligned}$$

which completes the proof. Now, let us give the following example.

Example 4.1: Let $a, c \in \mathbb{R}^+, b \in \mathbb{R}$. If

$$\begin{aligned} E(\xi_k^{(n)}) &= \frac{a}{n}, \quad E(\xi_k^{(n)})^{1+\delta} = o\left(\frac{1}{n}\right), \\ E(\eta_k^{(n)}) &= \frac{b}{n}, \quad \text{var}(\eta_k^{(n)}) = \frac{c}{n}, \quad E(\xi_k^{(n)})^{1+\delta} = o\left(\frac{1}{n}\right) \end{aligned}$$

for every $\delta > 0$, then

$$\sum_{k \leq nt} \xi_k^{(n)}(t) \rightarrow \xi(t) = a.t$$

on probability and

$$\sum_{k \leq nt} \eta_k^{(n)}(t) \rightarrow \eta(t) = b.t + \sqrt{\frac{c}{a}}.W(t)$$

on distribution, where $W(t)$ is Wiener process. Furthermore, it is obvious that $\xi^{-1} = t/a$. Thus the limit process $Y(t)$ will be as follows:

$$Y(t) = \eta(\xi^{-1}(t)) = (b/a)t + \sqrt{c}W(t/a) = (b/a)t + \sqrt{c/a}W(t)$$

And

$$Y(\gamma'(t)) = \eta(\xi^{-1}(\gamma'(t))) = (b/a)\gamma'(t) + \sqrt{c/a}W(\gamma'(t)).$$

We shall find a form of the process $\gamma'(t)$. The Process $\gamma(t)$ is the type of $\xi(t)$. If $E(\xi_k^{(n)}) = u/n$, then it is obvious that $u > a$. We known that $\gamma_1^{(n)} = \xi_1^{(n)} + \xi_2^{(n)} + \dots + \xi_{\gamma_1^{(n)}}^{(n)}$. Also $\gamma_1^{(n)}$ is the first moment of reaching to the zero-level of the process $\tilde{\chi}_n(t)$. The calculation of the expected value of random variable $\gamma_1^{(n)}$ is a very difficult problem of the stochastic process theory. Here we may take $u \in (0,1)$. Thus $\gamma'(t) = u.t$. Then we have

$$\begin{aligned} \tilde{\chi}(t) = Y(t) - Y(\gamma'(t)) &= (b/a)t + \sqrt{c/a}W(t) - (b/a)u.t - \sqrt{(c/a)u}W(t) \\ &= (b/a)(1-u)t + \sqrt{c/a}(1-\sqrt{u})W(t). \end{aligned}$$

Thus, we can write

$$\begin{aligned} \tilde{\chi}^*(t) &= \beta + Y(t) - Y(\gamma'(t)) - \sup_{0 \leq s \leq t} \{\beta, Y(s) - Y(\gamma'(s))\} \\ &= \beta + (b/a)(1-u)t + \sqrt{c/a}(1-\sqrt{u})W(t) \\ &\quad - \sup_{0 \leq s \leq t} \{\beta, (b/a)(1-u)s + \sqrt{c/a}(1-\sqrt{u})W(s)\}. \end{aligned}$$

5. CONCLUSION AND RECOMMENDATIONS (SONUÇ VE ÖNERİLER)



In this paper, a semi-Markovian random walk process with reflecting barrier on the zero-level and delaying barrier on the $\beta(\beta > 0)$ -level is constructed mathematically. Also it is given a sequence of this type processes within same way. Then it is proved some limit theorems for the sequence of this type processes which is convergences a semi-Markovian random walk process.

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